9.5 Homogeneous Linear Systems with Constant Coefficients

Theorem 5. *Suppose the $n \times n$ constant matrix A has n linearly independent eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Let λ_i be the eigenvalue corresponding to $\mathbf{u}_i$. Then*

$$\{e^{\lambda_1 t}\mathbf{u}_1, \ldots, e^{\lambda_n t}\mathbf{u}_n\} \tag{1}$$

is a fundamental solution set for the homogeneous system $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$. Consequently, a general solution of $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1(t) + \cdots + c_n e^{\lambda_n t}\mathbf{x}_n(t), \tag{2}$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Proof. The vector functions in (1) are solutions to the homogeneous system. Moreover, their Wronskian is

$$W(t) = \det\begin{bmatrix} e^{\lambda_1 t}\mathbf{u}_1 & \cdots & e^{\lambda_n t}\mathbf{u}_n \end{bmatrix} = e^{(\lambda_1 + \cdots + \lambda_n)t}\det\begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix}.$$

Since the eigenvectors are assumed to be linearly independent, it follows that $\det\begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix}$ is not zero. Hence the Wronskian $W(t)$ is never zero. This shows that (1) is a fundamental solution set, and consequently a general solution is given by (2). □

Theorem 6. *If $\lambda_1, \ldots, \lambda_m$ are* distinct *eigenvalues for the matrix A and $\mathbf{u}_i$ is an eigenvector associated with λ_i, then $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent.*

Corollary. *If the $n \times n$ constant matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mathbf{u}_i$ is an eigenvector associated with λ_i, then*

$$\{e^{\lambda_1 t}\mathbf{u}_1, \ldots, e^{\lambda_n t}\mathbf{u}_n\}$$

is a fundamental solution set for the homogeneous system $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$.

Remark. *A real $n \times n$ symmetric matrix* always *has n linearly independent eigenvectors and all its eigenvalues are real.*

Example 1. *Find a general solution of the system $\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}$ for the given matrix*

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}.$$

Example 2. *Consider the system $\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}, t \geq 0$, with $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.*

a) Show that the matrix A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_1$.

c) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = -\mathbf{u}_2$.

d) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_1 - \mathbf{u}_2$.

Example 3. *Find a fundamental matrix for the system $\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}$ for $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$.*

Example 4. *Solve the initial value problem*

$$\mathbf{x}'(t) - \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}\mathbf{x}(t) = \mathbf{0}, \quad \mathbf{x}(0) = \begin{bmatrix} -10 \\ -6 \end{bmatrix}.$$