9.4 Linear Systems in Normal Form

We say that a system of n linear differential equations is in *normal form* if it is expressed as

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{f}(t), \tag{1}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$, and $A(t) = [a_{ij}(t)]$ is an $n \times n$ matrix. The system is *homogeneous* when $\mathbf{f}(t) = 0$, otherwise the system is *nonhomogeneous*. An nth-order linear differential equation

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y(t) = g(t) \tag{2}$$

can be written as a first-order system in normal form using the substitution $x_1(t) = y(t), x_2(t) = y'(t), \ldots, x_n(t) = y^{(n-1)}(t)$. Equation (2) is equivalent to $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$, where $\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{bmatrix}$, and

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-2}(t) & -p_{n-1}(t) \end{bmatrix}.$$

The *initial value problem* for the normal system (1) is the problem of finding a differentiable vector function $\mathbf{x}(t)$ that satisfies the system on an interval I and also satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where t_0 is a given point of I and $\mathbf{x}_0 = \begin{bmatrix} x_{1,0} \\ \vdots \\ x_{n,0} \end{bmatrix}$ is a given vector.

Theorem 2 (Existence and Uniqueness). *If $A(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I that contains the point t_0, then for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ on the whole interval I to the initial value problem*

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

We may write system (1) as $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$. Let $L(\mathbf{x}) = \mathbf{x}' - A\mathbf{x}$. Then L is a linear operator that maps vector functions into vector functions.

Definition. *The m vector functions $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are* linearly dependent *on an interval I if there exist constants $c_1, \ldots, c_m$, not all zero, such that*

$$c_1\mathbf{x}_1(t) + \cdots + c_m\mathbf{x}_m(t) = \mathbf{0} \tag{3}$$

for all t in I. If the vectors are not linearly dependent, they are said to be linearly independent on I.

Definition. *The* Wronskian *of n vector functions $\mathbf{x}_1(t) = \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{n,1} \end{bmatrix}, \ldots, \mathbf{x}_n = \begin{bmatrix} x_{1,n} \\ \vdots \\ x_{n,n} \end{bmatrix}$ is defined to be the function*

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \begin{vmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{vmatrix}.$$

Remark. *Vector functions* $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ *are linearly independent on an interval if their Wronskian is nonzero* at any point *in the interval.*

Proposition. *If vector functions* $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ *are independent solutions to a homogeneous system* $L(\mathbf{x}) = \mathbf{0}$*, that is,* $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$*, where* A *is an* $n \times n$ *matrix of continuous functions, then the Wronskian is* never *zero on* I*.*

Proof. Suppose to the contrary that $W(t_0) = 0$. Then the column vectors $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)$ in the determinant are linearly dependent. Thus there exist scalars $c_1, \ldots, c_n$, *not all zero*, such that at $t = t_0$

$$c_1\mathbf{x}_1(t_0) + \cdots + c_n\mathbf{x}_n(t_0) = \mathbf{0}.$$

However, $c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ and the vector function $\mathbf{z}(t) = \mathbf{0}$ are both solutions to $L(\mathbf{x}) = \mathbf{0}$, and they agree at the point t_0. So these solutions must be identical on I according to the existence and uniqueness theorem. That is,

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0}$$

for *all* t in I. But this contradicts the given information that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent on I. Therefore $W(t_0) \neq 0$. Since t_0 is an arbitrary point, it follows that $W(t) \neq 0$ for all $t \in I$. □

Corollary. *The Wronskian of solutions to* $\mathbf{x}' = A\mathbf{x}$ *is either identically zero or never zero on* I*.*

Corollary. *A set of* n *solutions* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *to* $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$ *is linearly independent on* I *if and only if their Wronskian is never zero on* I*.*

Theorem 3 (Representation of Homogeneous Solutions)**.** *Let* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *be* n *linearly independent solutions to the homogeneous system*

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{0} \tag{4}$$

on the interval I*, where* $A(t)$ *is an* $n \times n$ *matrix function continuous on* I*. Then every solution to (4) on* I *can be expressed in the form*

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t), \tag{5}$$

where $c_1, \ldots, c_n$ *are constants.*

A set of linearly independent solutions $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, or equivalently, whose Wronskian does not vanish on I, is called a *fundamental solution set* for (4). The linear combination in (5), written with arbitrary constants, is called a *general solution* to (4).

A *fundamental matrix* for (4) is

$$X(t) = [\ \mathbf{x}_1(t) \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n(t)\ ] = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{bmatrix}.$$

Thus we may express the general solution (5) as

$$\mathbf{x}(t) = X(t)\mathbf{c},$$

where $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is an arbitrary constant vector. Since $\det X = W(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is never zero on I, it follows from Theorem 1 that $X(t)$ is invertible for every t in I. A corresponding *matrix differential equation* for $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$ is $X' - AX = 0$.

Since $L(\mathbf{x}) = \mathbf{x}' - A\mathbf{x}$ is a linear operator, the superposition principle for linear systems follows, that is, if $\mathbf{x}_1$ is a solution to $L(\mathbf{x}) = \mathbf{g}_1$ and $\mathbf{x}_2$ is a solution to $L(\mathbf{x}) = \mathbf{g}_2$, then $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is a solution to $L(\mathbf{x}) = c_1\mathbf{g}_1 + c_2\mathbf{g}_2$.

The following theorem follows from the superposition principle and the representation theorem for homogeneous systems.

Theorem 4. *If* $\mathbf{x}_p$ *is a particular solution to the nonhomogeneous system*

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{f}(t) \tag{6}$$

on the interval I *and* $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ *is a fundamental solution set on* I *for the corresponding homogeneous system* $\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{0}$*, then every solution to (6) on* I *can be expressed in the form*

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t), \tag{7}$$

where $c_1, \ldots, c_n$ *are constants.*

The linear combination in (7) is called a *general solution* of (6). We may write (7) as $\mathbf{x} = \mathbf{x}_p + X\mathbf{c}$, where X is a fundamental matrix for the homogeneous system and $\mathbf{c}$ is an arbitrary constant vector.

9.4.1 Approach to Solving Normal Systems

1. To determine a general solution to the $n \times n$ homogeneous system $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$:
   (a) Find a fundamental solution set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ that consists of n linearly independent solutions to the homogeneous system.
   (b) Form the linear combination
   $$\mathbf{x} = X\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n,$$
   where $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is any constant vector and $X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$ is the fundamental matrix, to obtain a general solution.

2. To determine a general solution to the nonhomogeneous system $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$:
   (a) Find a particular solution $\mathbf{x}_p$ to the nonhomogeneous system.
   (b) Form the sum of the particular solution and the general solution $X\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ to the corresponding homogeneous system in part 1,
   $$\mathbf{x} = \mathbf{x}_p + X\mathbf{c} = \mathbf{x}_p + c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n,$$
   to obtain a general solution to the nonhomogeneous system.

Example 1. *Write the given system in the matrix form* $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$*.*

$$\frac{dx}{dt} = x + y + z$$
$$\frac{dy}{dt} = 2x - y + 3z$$
$$\frac{dz}{dt} = x + 5z.$$

Example 2. *Rewrite the scalar equation as a first-order system in normal form. Express the system in the matrix form* $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$*.*

$$\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \cos t.$$

Example 3. *Determine whether the given vector functions are linearly independent or linearly dependent on the interval* $(-\infty, \infty)$*.*

a) $\begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix}, \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}.$

b) $\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \begin{bmatrix} \sin t \\ \sin t \end{bmatrix}, \begin{bmatrix} \cos t \\ \cos t \end{bmatrix}.$

Example 4. *The vector functions* $\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}$ *are solutions to a system* $\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}$*. Determine whether they form a fundamental solution set. If they do, find a fundamental matrix for the system and give a general solution.*