9 Matrix Methods for Linear Systems

9.1 Introduction

We may write a system of differential equations in matrix form. A system of differential equations is in *normal form* if the system is expressed as

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\
&\vdots \\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n.
\end{aligned}$$

The matrix form of such a system is $\mathbf{x}' = A\mathbf{x}$, where A is the coefficient matrix

$$A = A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

and $\mathbf{x}$ is the solution vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Example 1. *Express the given system of differential equations in matrix notation.*

$$\begin{aligned}
x_1' &= x_1 - x_2 + x_3 - x_4 \\
x_2' &= x_1 + x_4 \\
x_3' &= \sqrt{\pi}x_1 - x_3 \\
x_4' &= 0.
\end{aligned}$$

Example 2. *Express the Legendre's equation as a matrix system in normal form.*

$$(1 - t^2)y'' - 2ty' + 2y = 0.$$

Example 3. *Express the given system as a matrix system in normal form.*

$$\begin{aligned}
x'' + 3x' - y' + 2y &= 0 \\
y'' + x' + 3y' + y &= 0.
\end{aligned}$$