7 Symmetric Matrices

7.1 Diagonalization of Symmetric Matrices

A *symmetric* matrix is a matrix A such that $A^T = A$. Such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries occur in pairs, on opposite sides of the main diagonal.

Example 1. *Determine which matrix is symmetric.*

a) $\begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

Theorem 1. *If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.*

Proof. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that correspond to distinct eigenvalues, say λ_1 and λ_2. To show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, compute

$$\begin{aligned} \lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 &= (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 \\ &= (A\mathbf{v}_1)^T \mathbf{v}_2 \\ &= (\mathbf{v}_1^T A^T)\mathbf{v}_2 \\ &= \mathbf{v}_1^T (A\mathbf{v}_2) \\ &= \mathbf{v}_1^T (\lambda_2 \mathbf{v}_2) \\ &= \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \\ &= \lambda_2 \mathbf{v} \cdot \mathbf{v}_2. \end{aligned}$$

Hence $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. But $\lambda_1 - \lambda_2 \neq 0$, so $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. □

An $n \times n$ matrix A is said to be *orthogonally diagonalizable* if there are an orthogonal matrix P (with $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1}. \tag{1}$$

Such a diagonalization requires n linearly independent and orthonormal eigenvectors. When is this possible? If A is orthogonally diagonalizable as in (1), then

$$A^T = (PDP^T)^{=} P^{TT} D^T P^T = PDP^T = A.$$

Thus A is symmetric!

Theorem 2. *An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.*

The set of eigenvalues of a matrix A is sometimes called the *spectrum* of A.

Theorem 3 (The Spectral Theorem for Symmetric Matrices)**.** *An $n \times n$ symmetric matrix A has the following properties:*

1. *A has n real eigenvalues, counting multiplicities.*

2. *The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.*

3. *The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.*

4. *A is orthogonally diagonalizable.*

7.1.1 Spectral Decomposition

Suppose $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ of A and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are in the diagonal matrix D. Then, since $P^{-1} = P^T$,

$$\begin{aligned} A &= PDP^T \\ &= [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \\ &= [\lambda_1 \mathbf{u}_1 \quad \cdots \quad \lambda_n \mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}. \end{aligned}$$

Using the column-row expansion of a product, we can write

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T. \tag{2}$$

This representation of A is called a *spectral decomposition* of A because it breaks up A into pieces determined by the spectrum (eigenvalues) of A. Each term in (2) is an $n \times n$ matrix of rank 1. For example, every column of $\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T$ is a multiple of $\mathbf{u}_1$. Furthermore, each matrix $\mathbf{u}_j \mathbf{u}_j^T$ is a *projection matrix* in the sense that for each $\mathbf{x} \in \mathbb{R}^n$, the vector $(\mathbf{u}_j \mathbf{u}_j^T)\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto the subspace spanned by $\mathbf{u}_j$.

Example 2. *Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.*

a) Given any $\mathbf{x} \in \mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$.

b) Show that B is a symmetric matrix and $B^2 = B$.

c) Show that $\mathbf{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

Example 3. *Find the inverse of each orthogonal matrix.*

a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \end{bmatrix}$

Example 4. *Orthogonally diagonalize the matrix, giving an orthogonal matrix P and a diagonal matrix D. The eigenvalues are $-3, -6, 9$.*

$$\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$$