6.7 Cauchy-Schwarz Inequality

Recall that we may write a vector $\mathbf{u}$ as a scalar multiple of a nonzero vector $\mathbf{v}$, plus a vector orthogonal to $\mathbf{v}$:

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} + \left(\mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}\right). \tag{1}$$

The equation (1) will be used in the proof of the next theorem, which gives one of the most important inequalities in mathematics.

Theorem 16 (Cauchy-Schwarz Inequality)**.** *If* $\mathbf{u}, \mathbf{v} \in V$, *then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|. \tag{2}$$

This inequality is an equality if and only if one of $\mathbf{u}, \mathbf{v}$ *is a scalar multiple of the other.*

Proof. Let $\mathbf{u}, \mathbf{v} \in V$. If $\mathbf{v} = \mathbf{0}$, then both sides of (2) equal 0 and the desired inequality holds. Thus we can assume that $\mathbf{v} \neq \mathbf{0}$. Consider the orthogonal decomposition

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} + \mathbf{w},$$

where $\mathbf{w}$ is orthogonal to $\mathbf{v}$ (here $\mathbf{w}$ equals the second term on the right side of (1)). By the Pythagorean theorem,

$$\begin{aligned}
\|\mathbf{u}\|^2 &= \left\|\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}\right\|^2 + \|\mathbf{w}\|^2 \\
&= \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{v}\|^2} + \|\mathbf{w}\|^2 \\
&\geq \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{v}\|^2}.
\end{aligned}$$

Multiplying both sides of this inequality by $\|\mathbf{v}\|^2$ and then taking square roots gives the Cauchy-Schwarz inequality (2).

Looking at the proof of the Cauchy-Schwarz inequality, note that (2) is an equality if and only if the last inequality above is an equality. Obviously this happens if and only if $\mathbf{w} = \mathbf{0}$. But $\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{u}$ is a multiple of $\mathbf{v}$. Thus the Cauchy-Schwarz inequality is an equality if and only if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ or $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$ (or both; the phrasing has been chosen to cover cases in which either $\mathbf{u}$ or $\mathbf{v}$ equals $\mathbf{0}$). □

The next result is called the triangle inequality because of its geometric interpretation that the length of any side of a triangle is less than the sum of the lengths of the other two sides. Consider a triangle with sides consisting of vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} + \mathbf{v}$.

Theorem 17 (Triangle Inequality)**.** *If* $\mathbf{u}, \mathbf{v} \in V$, *then*

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|. \tag{3}$$

This inequality is an equality if and only if one of $\mathbf{u}, \mathbf{v}$ *is a nonnegative multiple of the other.*

Proof. Let $\mathbf{u}, \mathbf{v} \in V$. Then

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle \\
&\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle| \\
&\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Taking square roots of both sides of the inequality above gives the triangle inequality (3).

We have equality in the triangle inequality if and only if

$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\|. \tag{4}$$

If one of $\mathbf{u}, \mathbf{v}$ is a nonnegative multiple of the other, then (4) holds. Conversely, suppose (4) holds.Then the condition for equality in the Cauchy-Schwarz inequality implies that one of $\mathbf{u}, \mathbf{v}$ must be a scalar multiple of the other. Clearly (4) forces the scalar in question to be nonnegative, as desired. □

Example 1. *Suppose $p(t) = 3t - t^2$ and $q(t) = 3 + 2t^2$. For p and q in $\mathbb{P}_2$, define*

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

a) Compute $\langle p, q \rangle$.

b) Compute the orthogonal projection of q onto the subspace spanned by p.

Example 2. *Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use the Cauchy-Schwarz inequality to show that*

$$\left(\frac{a+b}{2} \right)^2 \leq \frac{a^2+b^2}{2}.$$