6.2 Orthogonal Sets

We say a set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an *orthogonal set* if each pair of distinct vectors from the set is orthogonal. That is, if $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

A list of vectors is called *orthonormal* if the vectors in it are pairwise orthogonal and each vector has norm 1. In other words, a list $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ of vectors in V is orthonormal if $\langle \mathbf{e}_j, \mathbf{e}_k \rangle$ equals 0 when $j \neq k$ and equals 1 when $j = k$, for $j, k = 1, \ldots, m$. For example, the standard basis in $\mathbb{R}^n$ is orthonormal. Orthonormal lists are particularly easy to work with, as illustrated by the next proposition.

Proposition. *If $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is an orthonormal list of vectors in V, then*

$$\|a_1\mathbf{e}_1 + \cdots + a_m\mathbf{e}_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{R}$.

Proof. Because each $\mathbf{e}_j$ has norm 1, this follows easily from repeated applications of the Pythagorean theorem. □

Corollary. *Every orthonormal list of vectors is linearly independent.*

Proof. Suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is an orthonormal list of vectors in V and $a_1, \ldots, a_m \in \mathbb{R}$ are such that

$$a_1\mathbf{e}_1 + \cdots + a_m\mathbf{e}_m = 0.$$

Then $|a_1|^2 + \cdots + |a_m|^2 = 0$ by the previous proposition, which means that all a_j's are 0, as desired. □

An *orthonormal basis* of V is an orthonormal list of vectors in V that is also a basis of V. For example, the standard basis is an orthonormal basis of $\mathbb{R}^n$. Every orthonormal list of vectors in V with length dim V is automatically an orthonormal basis of V (proof: by the previous corollary, any such list must be linearly independent; because it has the right length, it must be a basis). An orthogonal basis is a basis that is also an orthogonal set.

Example 1. *Consider the following list of four vectors in $\mathbb{R}^4$:*

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \right\}$$

The verification that this list is orthonormal is easy. Because we have an orthonormal list of length four in a four-dimensional vector space, it must be an orthonormal basis.

Theorem 4. *If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in V, then S is linearly independent and hence is a basis for the subspace spanned by S.*

Theorem 5. *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a vector space W. For each $\mathbf{y} \in W$, the weights in the linear combination*

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\langle \mathbf{y}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \quad , j = 1, \ldots, p.$$

Proof. The orthogonality of $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ shows that

$$\langle \mathbf{y}, \mathbf{u}_j \rangle = \langle (c_1\mathbf{u}_1 + \cdots + c_j\mathbf{u}_j + \cdots + c_p\mathbf{u}_p), \mathbf{u}_p \rangle = c_j \langle \mathbf{u}_j, \mathbf{u}_j \rangle.$$

Since $\langle \mathbf{u}_j, \mathbf{u}_j \rangle \neq 0$, the equation above has a solution for c_j. □

In general, given a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V and a vector $\mathbf{v} \in V$, we know that there is a unique choice of scalars $a_1, \ldots, a_n$ such that

$$\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n,$$

but finding the a_j's can be difficult. The next theorem shows, however, that this is easy for an orthonormal basis.

Theorem. *Suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis of V. Then*

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{e}_n \tag{1}$$

and

$$\|\mathbf{v}\|^2 = |\langle \mathbf{v}, \mathbf{e}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{e}_n \rangle|^2 \tag{2}$$

for every $v \in V$.

Proof. Let $\mathbf{v} \in V$. Because $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of V, there exist scalars $a_1, \ldots, a_n$ such that

$$\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n.$$

Take the inner product of both sides of this equation with $\mathbf{e}_j$, getting $\langle \mathbf{v}, \mathbf{e}_j \rangle = a_j$. Thus (1) holds. Clearly (2) follows from (1) and the first proposition on orthonormal lists. $\square$

Theorem 6. *An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.*

Theorem 7. *Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then*

1. $\|U\mathbf{x}\| = \|\mathbf{x}\|$
2. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0 \Leftrightarrow \mathbf{x} \cdot \mathbf{y} = 0$.

An *orthogonal matrix* is a square invertible matrix U such that $U^{-1} = U^T$. By Theorem 6, such a matrix has orthonormal columns. It is easy to see that any square matrix with orthonormal columns is an orthogonal matrix. Surprisingly, such a matrix must have orthonormal rows, too.

Example 2. *Determine if the following set is orthogonal.*

$$\left\{ \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \right\}$$

Example 3. *Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$ and write $\mathbf{x} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$.*