5.4 Eigenvectors and Linear Transformations

Suppose $T : V \to W$ is a linear map with an $m \times n$ matrix A. Choose ordered bases $\mathcal{B}$ for n-dimensional vector space V and $\mathcal{C}$ for m-dimensional vector space W. Given every vector $\mathbf{x} \in V$, the coordinate vector $[\mathbf{x}]_\mathcal{B}$ is in $\mathbb{R}^n$, and the coordinate vector of its image, $[T\mathbf{x}]_\mathcal{C}$, is in $\mathbb{R}^m$. The connection between $[\mathbf{x}]_\mathcal{B}$ and $[T\mathbf{x}]_\mathcal{C}$ is easy to find. Let $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be the basis $\mathcal{B}$ for V. If $\mathbf{x} = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n$, then $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$ and

$$T\mathbf{x} = T(r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n) = r_1 T\mathbf{b}_1 + \cdots + r_n T\mathbf{b}_n \tag{1}$$

because T is linear. Now, since the coordinate mapping from W to $\mathbb{R}^m$ is linear, equation (1) leads to

$$[T\mathbf{x}]_\mathcal{C} = r_1[T\mathbf{b}_1]_\mathcal{C} + \cdots + r_n[T\mathbf{b}_n]_\mathcal{C} \tag{2}$$

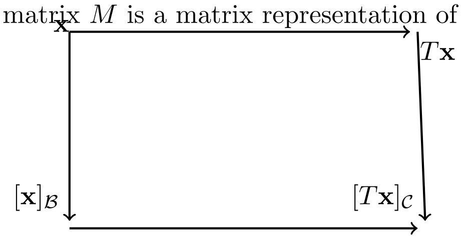
Since the coordinate vectors are in $\mathbb{R}^m$, we may write the vector equation (2) as a matrix equation:

$$[T\mathbf{x}]_\mathcal{C} = M[\mathbf{x}]_\mathcal{B} \tag{3}$$

where

$$M = [[T\mathbf{b}_1]_\mathcal{C} \quad \cdots [T\mathbf{b}_n]_\mathcal{C}]\,. \tag{4}$$

The matrix M is a matrix representation of T, called the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.



In the common case where $: V \to V$ and the bases $\mathcal{B} = \mathcal{C}$, then the matrix M in (4) is called the matrix for T *relative to* $\mathcal{B}$, or simply the $\mathcal{B}$-matrix for T, and is denoted by $[T]_\mathcal{B}$.

Theorem 8 (Diagonal Matrix Representation)**.** *Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed by the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.*

Example 1. *Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V and $T : V \to \mathbb{R}^2$ be a linear transformation with the property that*

$$T(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}.$$

Find the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$.

Example 2. *Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $\mathbf{p}(t) + t^2\mathbf{p}(t)$.*

a) Find the image of $\mathbf{p}(t) = 2 - t + t^2$.

b) Show that T is a linear map.

c) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

Example 3. *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\mathbf{x} = A\mathbf{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_\mathcal{B}$ is diagonal.*

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}.$$