5.3 Diagonalization

A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.

Example 1. *The matrix* $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ *is a diagonal matrix.*

Obviously every diagonal matrix is a triangular matrix, although in general a diagonal matrix has many more 0's than a triangular matrix. In general, the more zeros a matrix has, the easier it is to do calculations on that matrix. For instance, it is easy to calculate powers of a diagonal matrix: just raise each entry on the diagonal to the desired exponent.

Unfortunately not all matrices are diagonal. However, certain matrices are similar to a diagonal matrix.

Definition. *A square matrix A is said to be* diagonalizable *if A is similar to a diagonal matrix, that is, if $A = P^{-1}DP$ for some invertible matrix P and some diagonal matrix D.*

Theorem 5 (The Diagonalization Theorem)**.** *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = P^{-1}DP$, with D a diagonal matrix, if and only if the columns of P are linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an *eigenvector basis* of $\mathbb{R}^n$.

Example 2. *Suppose* $A = PDP^{-1}$*, where* $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}, P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$.

1. *Find A^k for arbitrary positive integer k.*

2. *Find eigenvalues of A and a basis for each eigenspace.*

Example 3. *Find the eigenvalues of $A = PDP^{-1}$ and a basis for each eigenspace.*

$$\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

There are four steps to diagonalize an $n \times n$ matrix $A = PDP^{-1}$:

1. Find the eigenvalues of A.

2. Find n linearly independent eigenvectors of A.

3. Construct P from the vectors in step 2.

4. Construct D from the corresponding eigenvalues.

Example 4. *Diagonalize the following matrices, if possible.*

1. $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

2. $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

Theorem 6. *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

When A is diagonalizable but has fewer than n distinct eigenvalues, it is still possible to build P in a way that makes P automatically invertible, as the next theorem shows.

Theorem 7. *Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.*

1. *For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.*
2. *The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.*
3. *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*

Proposition. *Suppose T is an operator on V. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T with matrix A. The the following are equivalent:*

1. *T has a diagonal matrix with respect to some basis of V;*
2. *V has a basis consisting of eigenvectors of T;*
3. $\dim V = \dim \text{Nul}(A - \lambda_1 I) + \cdots + \dim \text{Nul}(A - \lambda_m I)$.