5 Eigenvalues and Eigenvectors

5.1 Eigenvectors and Eigenvalues

An *operator* is a linear map from a vector space to itself. The notion of a subspace that gets mapped into itself is sufficiently important to deserve a name. Thus, for T an operator from V into V and U a subspace of V, we say that U is *invariant* under T if $\mathbf{u} \in U$ implies $T\mathbf{u} \in U$. For example, if T is the operator of differentiation on $\mathbb{P}_7$, then $\mathbb{P}_4$, which is a subspace of $\mathbb{P}_7$, is invariant under T because the derivative of any polynomial of degree at most 4 is also a polynomial with degree at most 4.

How does an operator behave on an invariant subspace of dimension 1? Subspaces of V of dimension 1 are easy to describe. Take any nonzero vector $\mathbf{u} \in V$ and let U equal the set of all scalar multiples of $\mathbf{u}$:

$$U = \{a\mathbf{u} : a \in \mathbb{R}\}. \tag{1}$$

Then U is a one-dimensional subspace of V, and every one-dimensional subspace of V is of this form. If $\mathbf{u} \in V$ and the subspace U defined by (1) is invariant under $T : V \to V$, then $T\mathbf{u}$ must be in U, and hence there must be a scalar $\lambda \in \mathbb{R}$ such that $T\mathbf{u} = \lambda u$. Converseley, if $\mathbf{u}$ is a nonzero vector in V such that $T\mathbf{u} = \lambda u$ for some $\lambda \in \mathbb{R}$, then the subspace U defined by (1) is a one-dimensional subspace of V, invariant under T.

The equation

$$T\mathbf{u} = \lambda\mathbf{u} \tag{2}$$

which we have just seen is intimately connected with one-dimensional invariant subspaces, is important enough that the vectors $\mathbf{u}$ and scalars λ satisfying it are given special names.

Definition. *A scalar $\lambda \in \mathbb{R}$ is called an* eigenvalue *of operator $T : V \to V$ if there exists a nonzero vector $\mathbf{u} \in V$ such that $T\mathbf{u} = \lambda\mathbf{u}$.*

We must require $\mathbf{u}$ to be nonzero because with $\mathbf{u} = \mathbf{0}$, every scalar $\lambda \in \mathbb{R}$ satisfies (2). The comments above show that T has a one-dimensional invariant subspace if and only if T has an eigenvalue. If the standard matrix for T is A, then we have $T\mathbf{x} = A\mathbf{x}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ is equivalent to $(A - \lambda I)\mathbf{x} = \mathbf{0}$. So λ is an eigenvalue of T if and only if $A - \lambda I$ is not invertible.

Definition. *Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of A. A vector $\mathbf{x} \in V$ is called an* eigenvector *of A (corresponding to λ) if $A\mathbf{x} = \lambda\mathbf{x}$.*

Because (2) is equivalent to $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we see that the set of eigenvectors of A corresponding to λ equals $\mathrm{Nul}(A - \lambda I)$. In particular, the set of eigenvectors of A corresponding to A is a subspace of V.

Note that λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

has a nontrivial solution. The set of all solutions of (3) is just the null space of matrix $A - \lambda I$. So this set is a subspace of $\mathbb{R}^n$ and is called the *eigenspace* of A corresponding to λ.

Theorem 1. *Suppose $T : V \to V$ has an upper-triangular matrix with respect to some basis of V. The the eigenvalues of T consist precisely of the entries on the diagonal of that upper-triangular matrix.*

Proof. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V with respect to which T has an upper-triangular matrix

$$A = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

Let $\lambda \in \mathbb{R}$. Then

$$A - \lambda I = \begin{bmatrix} \lambda_1 - \lambda & & & * \\ & \lambda_2 - \lambda & & \\ & & \ddots & \\ 0 & & & \lambda_n - \lambda \end{bmatrix}.$$

Hence $A - \lambda I$ is not invertible if and only if λ equals one of the λ_j's. In other words, λ is an eigenvalue of T if and only if λ equals one of the λ_j's, as desired. ☐

Theorem 2. *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.*

Proof. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent. Let k be the smallest positive integer such that

$$\mathbf{v}_k \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}. \tag{4}$$

Thus there exist $a_1, \ldots, a_{k-1} \in \mathbb{R}$ such that

$$\mathbf{v}_k = a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}. \tag{5}$$

Apply A to both sides of this equation, getting

$$\lambda_k\mathbf{v}_k = a_1\lambda_1\mathbf{v}_1 + \cdots + a_{k-1}\lambda_{k-1}\mathbf{v}_{k-1}.$$

Multiply both sides of (5) by λ_k and then subtract the equation above, getting

$$\mathbf{0} = a_1(\lambda_k - \lambda_1)\mathbf{v}_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1})\mathbf{v}_{k-1}.$$

Because we chose k to be the smallest positive integer satisfying (4), $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$ is linearly independent. Thus the equation above implies that all a's are 0 (recall that λ_k is not equal to any of $\lambda_1, \ldots, \lambda_{k-1}$). However, this means that $\mathbf{v}_k$ equals $\mathbf{0}$ (see (5)), contradicting our hypothesis that all $\mathbf{v}$'s are nonzero. Therefore our assumption that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent must have been false. ☐

Corollary. *Each operator on V has at most* dim V *distinct eigenvalues.*

Proof. Let $T : V \to V$ be an operator. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be corresponding nonzero eigenvectors. The last theorem implies that $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent. Thus $m \leq \dim V$, as desired. ☐

Example 1. *Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.*

Example 2. *Find a basis for the eigenspace corresponding to each listed eigenvalue.*

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \lambda = 1, 5$$

Example 3. *Find the eigenvalues of the following matrix.*

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

Example 4. *Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Justify.*

Example 5. *Suppose A is an $n \times n$ matrix. Mark each statement True or False. Justify.*

1. *If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.*
2. *If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.*
3. *The eigenvalues of a matrix are on its main diagonal.*
4. *An eigenspace of A is a null space of a certain matrix.*

Example 6. *Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [Hint: Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]*

Example 7. *Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. [Hint: Find an eigenvector.]*

Example 8. *Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [Hint: Use the previous two examples.]*