10.5 The Heat Equation

A mathematical model for source-less the heat flow in a uniform wire whose ends are kept at constant temperature 0 is the following initial value problem, where $u(x,t)$ is the temperature in the wire at location x and time t:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0, \tag{1}$$

$$u(0,t) = u(L,t) = 0, \quad t \geq 0, \tag{2}$$

$$u(x,0) = f(x), \quad 0 < x < L. \tag{3}$$

Equation (2) specifies that the temperature at the ends of the wire is zero. Equation (3) specifies the initial temperature distribution. Using the method of separation of parameters, we may find the solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L}. \tag{4}$$

where the c_n's are the coefficients in the Fourier sine series for $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}. \tag{5}$$

In other words, solving (1)-(3) reduces to computing the Fourier sine series for the initial value function $f(x)$.

In this section, we discuss heat flow problems where the ends of the wire are kept at a constant temperature other than zero, that is, nonhomogeneous boundary conditions. We will also discuss the problem in which a heat source is adding heat to the wire, that is, a nonhomogeneous partial differential equation. The problem of heat flow in a rectangular plate leads to the topic of double Fourier series.

Suppose the ends of the wire are insulated, that is, no heat flows in or out at the ends. It follows from the principle of heat conduction that the temperature gradient must be zero at the endpoints, that is,

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0.$$

Example 1. *Find a formal solution to the heat flow problem governed by the initial-boundary value problem*

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, t > 0 \tag{6}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \qquad t > 0 \tag{7}$$

$$u(x,0) = f(x), \qquad 0 < x < L. \tag{8}$$

Solution. Using the method of separation of variables, we first assume that

$$u(x,t) = X(t)T(t).$$

Substituting into (6) and separating variables, we get two equations

$$X''(x) + \lambda X(x) = 0 \tag{9}$$

$$T'(t) + \beta\lambda T(t) = 0, \tag{10}$$

where λ is some constant. The boundary conditions in (7) become

$$X'(0)T(t) = 0 \quad \text{and} \quad X'(L)T(t) = 0.$$

For these equations to hold for all $t0$, either $T(t) = 0$, which implies that $u(x,t) = 0$, or

$$X'(0) = X'(L) = 0. \tag{11}$$

Combining the boundary conditions in (11) with (9) gives the boundary value problem

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(L) = 0, \tag{12}$$

where λ can be any constant.

To solve for the nontrivial solutions to (12), we try $X(x) = e^{rx}$ and form the auxiliary equation $r^2 + \lambda = 0$. When $\lambda < 0$, there are no trivial solutions to (12).

When $\lambda = 0$, the auxiliary equation has the repeated root 0 and a general solution to the differential equation is

$$X(x) = A + Bx.$$

The boundary conditions in (12) reduce to $B = 0$ with A arbitrary. Thus, for $\lambda = 0$, the nontrivial solutions to (12) are of the form

$$X(x) = c_0,$$

where $c_0 = A$ is an arbitrary nonzero constant.

When $\lambda > 0$, the auxiliary equation has the roots $r = \pm i\sqrt{\lambda}$. Thus, a general solution to the differential equation in (12) is

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

The boundary conditions in (12) lead to the system

$$\begin{aligned} \sqrt{\lambda}C_2 &= 0 \\ -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}L + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}L &= 0. \end{aligned}$$

Hence, $C_2 = 0$ and the system reduces to solving $c_1 \sin\sqrt{\lambda}L = 0$. Since $\sin\sqrt{\lambda}L = 0$ only when $\sqrt{\lambda}L = n\pi$, where n is an integer, we obtain a nontrivial solution only when $\sqrt{\lambda} = n\pi/L$ or $\lambda = (n\pi/L)^2, n = 1, 2, 3, \ldots$. Furthermore, the nontrivial solutions (eigenfunctions) X_n corresponding to the eigenvalue $\lambda = (n\pi/L)^2$ are given by

$$X_n(x) = c_n \cos\frac{n\pi x}{L}, \tag{13}$$

where the c_n's are arbitrary nonzero constants. In fact, formula (13) also holds for $n = 0$, since $\lambda = 0$ has the eigenfunction $X_0(x) = c_0$.

Having determined that $\lambda = (n\pi/L)^2, n = 0, 1, 2, \ldots$, let's consider (10) for such λ:

$$T'(t) + \beta(n\pi/L)^2 T(t) = 0.$$

For $n = 0, 1, 2, \ldots$, the general solution is

$$T_n(t) = b_n e^{-\beta(n\pi/L)^2 t},$$

where the b_n's are arbitrary constants. Combining this with equation (13), we obtain the functions

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) = \left[c_n \cos\frac{n\pi x}{L}\right]\left[b_n e^{-\beta(n\pi/L)^2 t}\right], \\ u_n(x,t) &= a_n e^{-\beta(n\pi/L)^2 t} \cos\frac{n\pi x}{L}, \end{aligned}$$

where $a_n = b_n c_n$ is, again, an arbitrary constant.

If we take an infinite series of these functions, we obtain

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta(n\pi/L)^2 t} \cos\frac{n\pi x}{L}, \tag{14}$$

which will be a solution to (6)-(7) provided the series has the proper convergence behavior. Notice that in (14) we have altered the constant term and written it as $a_0/2$, thus producing the standard form for cosine expansions.

Assuming a solution to (6)-(7) is given by the series in (14) and substituting into the initial conditions (8), we get

$$u(x,0)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}=f(x),\quad 0<x<L. \tag{15}$$

This means that if we choose the a_n's as the coefficients in the Fourier cosine series for f,

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx,\quad n=0,1,2,\ldots,$$

then $u(x,t)$ given in (14) will be a *formal solution* to the heat flow problem (6)-(8). Again, if this expansion converges to a continuous function with continuous second partial derivatives, then the formal solution is an actual solution. $\square$

Example 2. *Find a formal solution to the following initial-boundary value problem:*

$$\begin{aligned}\frac{\partial u}{\partial t}&=2\frac{\partial^2 u}{\partial x^2}, && 0<x<1, t>0\\ \frac{\partial u}{\partial x}(0,t)&=\frac{\partial u}{\partial x}(1,t)=0, && t>0\\ u(x,0)&=x(1-x), && 0<x<1.\end{aligned}$$

When the ends of the wire are kept at 0°C or when the ends are insulated, the boundary conditions are said to be *homogeneous*. But when the ends of the wire are kept at constant temperatures different from zero, that is,

$$u(0,t)=U_1\quad\text{and}\quad u(L,t)=U_2,\quad t>0, \tag{16}$$

then the boundary conditions are called *nonhomogeneous*.

We expect that the solution to the heat flow problem with nonhomogeneous boundary condition will consist of a *steady-state solution* $v(x)$ that satisfies the nonhomogeneous boundary conditions in (16) plus a *transient solution* $w(x,t)$. That is,

$$u(x,t)=v(x)+w(x,t), \tag{17}$$

where $w(x,t)$ and its partial derivatives tend to zero as $t\to\infty$. The function $w(x,t)$ will then satisfy the homogeneous boundary conditions.

Example 3. *Find a formal solution to the heat flow problem governed by the initial-boundary value problem*

$$\frac{\partial u}{\partial t}=\beta\frac{\partial^2 u}{\partial x^2},\qquad 0<x<L, t>0 \tag{18}$$

$$u(0,t)=U_1, u(L,t)=U_2,\qquad t>0 \tag{19}$$

$$u(x,0)=f(x),\qquad 0<x<L. \tag{20}$$

Solution. Suppose $u(x,t)$ satisfies (17). Substituting in equations (18)-(20) leads to

$$\frac{\partial u}{\partial t}=\frac{\partial w}{\partial t}=\beta v''(x)+\beta\frac{\partial^2 w}{\partial x^2},\qquad 0<x<L, t>0 \tag{21}$$

$$v(0)+w(0,t)=U_1, v(L)+w(L,t)=U_2,\qquad t>0 \tag{22}$$

$$v(x)+w(x,0)=f(x),\qquad 0<x<L. \tag{23}$$

If we allow $t \to \infty$ in (21)-(22), assuming that $w(x,t)$ is a transient solution, we obtain the steady-state boundary value problem

$$\begin{aligned} v''(x) &= 0, && 0 < x < L, \\ v(0) &= U_1, && v(L) = U_2. \end{aligned}$$

Solving for v, we obtain $v(x) = Ax + B$, and choosing A and B so that the boundary conditions are satisfied, yields

$$v(x) = U_1 + \frac{U_2 - U_1}{L} x \tag{24}$$

as the steady-state solution.

With this choice for $v(x)$, the initial-boundary value problem (21)-(23) reduces to the following initial-boundary value problem for $w(x,t)$:

$$\frac{\partial w}{\partial t} = \beta \frac{\partial^2 w}{\partial x^2}, \qquad 0 < x < L, t0 \tag{25}$$

$$w(0,t) = w(L,t) = 0, \qquad t > 0 \tag{26}$$

$$w(x,0) = f(x) - U_1 - \frac{U_2 - U_1}{L} x, \qquad 0 < x < L. \tag{27}$$

A formal solution to (25)-(27) is given by (4). Hence,

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L},$$

where the c_n's are the coefficients of the Fourier sine series expansion

$$f(x) - U_1 - \frac{U_2 - U_1}{L} x = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}.$$

Therefore, the formal solution to (18)-(20) is

$$u(x,t) = U_1 + \frac{U_2 - U_1}{L} x + \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L}, \tag{28}$$

with

$$c_n = \frac{2}{L} \int_0^L \left(f(x) - U_1 - \frac{U_2 - U_1}{L} x \right) \sin \frac{n\pi x}{L} dx.$$

□

The method of separation of variables is also applicable to problems in higher dimensions. For example, consider the problem of heat flow in a rectangular plate with sides $x = 0, x = L, y = 0$, and $y = W$. If the two sides $y = 0, y = W$ are kept at a constant temperature of 0°C and the two sides $x = 0, x = L$ are perfectly insulated, then heat flow is governed by the initial-boundary value problem in the following example.

Example 4. *Find a formal solution $u(x,y,t)$ to the initial-boundary value problem*

$$\frac{\partial u}{\partial t} = \beta \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^u}{\partial y^2} \right\}, \qquad 0 < x < L, 0 < y < W, t > 0 \tag{29}$$

$$\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(L,y,t) = 0, \qquad 0 < y < W, t > 0 \tag{30}$$

$$u(x,0,t) = u(x,W,t) = 0, \qquad 0 < x < L, t > 0 \tag{31}$$

$$u(x,y,0) = f(x,y), \qquad 0 < x < L, 0 < y < W. \tag{32}$$

Solution. If we assume a solution of the form $u(x, y, t) = V(x, y)T(t)$, then equation (29) separates into the two equations

$$T''(t) + \beta\lambda T(t) = 0 \tag{33}$$

$$\frac{\partial^2 V}{\partial x^2}(x, y) + \frac{\partial^2 V}{\partial y^2}(x, y) + \lambda V(x, y) = 0, \tag{34}$$

where λ can be any constant. To solve equation (34), we again use separation of variables. Here we assume $V(x, y) = X(x)Y(y)$. This allows us to separate equation (34) into the two equations

$$X''(x) + \mu X(x) = 0 \tag{35}$$

$$Y''(y) + (\lambda - \mu)Y(y) = 0, \tag{36}$$

where μ can be any constant. To solve for $X(x)$, we observe that the boundary conditions in (30), in terms of the separated variables, become

$$X'(0)Y(y)T9t) = X'(L)Y(y)T(t) = 0, \quad 0 < y < W, t > 0.$$

Here, in order to get a nontrivial solution, we must have

$$X'(0) = X'(L) = 0. \tag{37}$$

The boundary value problem for X given in (35) and (37) was solved in a previous example. Here $\mu = (m\pi/L)^2, m = 0, 1, 2, \ldots$, and

$$X_m(x) = c_m \cos \frac{m\pi x}{L},$$

where the c_m's are arbitrary.

To solve for $Y(y)$, we first observe that the boundary conditions in (31) become

$$Y(0) = Y(W) = 0. \tag{38}$$

Next, substituting $\mu = (m\pi/L)^2$ into (36) yields

$$Y''(y) + (\lambda - (m\pi/L)^2)Y(y) = 0,$$

which we can rewrite as

$$Y''(y) + EY(y) = 0, \tag{39}$$

where $E = \lambda - (m\pi/L)^2$. It can be shown that $E = (n\pi/W)^2, n = 1, 2, 3, \ldots$ and the nontrivial solutions of the boundary value problem for Y consisting of (38)-(39) are

$$Y_n(y) = a_n \sin \frac{n\pi y}{W},$$

where the a_n's are arbitrary.

Since $\lambda = E + (m\pi/L)^2$, we have

$$\lambda = (n\pi/W)^2 + (m\pi/L)^2, \quad m = 0, 1, 2, \ldots, \quad n = 1, 2, 3, \ldots.$$

Substituting λ into (33), we can solve for $T(t)$ and obtain

$$T_{mn}(t) = b_{mn} e^{-(m^2/L^2 + n^2/W^2)\beta\pi^2 t}.$$

Substituting in for X_m, Y_n, and T_{mn}, we get

$$u_{mn}(x, y, t) = \left(c_m \cos \frac{m\pi x}{L}\right)\left(a_n \sin \frac{n\pi y}{W}\right)\left(b_{mn} e^{-(m^2/L^2 + n^2/W^2)\beta\pi^2 t}\right) \tag{40}$$

$$u_{mn}(x, y, t) = a_{mn} e^{-(m^2/L^2 + n^2/W^2)\beta\pi^2 t} \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{W}, \tag{41}$$

where $a_{mn} = a_n b_{mn} c_m$ $(m = 0, 1, 2, \ldots, n = 1, 2, 3, \ldots)$ are arbitrary constants.

If we now take a doubly infinite series of such functions, then we obtain the formal series

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} e^{-(m^2/L^2 + n^2/W^2)\beta\pi^2 t} \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{W}. \tag{42}$$

We are now ready to apply the initial conditions (32). Setting $t = 0$, we obtain

$$u(x, y, 0) = f(x, y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{W}. \tag{43}$$

This is a *double Fourier series.* The formula for the coefficients a_{mn} are obtained by exploiting the orthogonality conditions twice. Presuming (43) is valid and permits term-by-term integration, we multiply each side by $\cos(p\pi x/L)\sin(q\pi y/W)$ and integrate over x and y:

$$\int_0^L \int_0^W f(x, y) \cos \frac{p\pi x}{L} \sin \frac{q\pi y}{W} dy dx = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \int_0^L \int_0^W \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{W} \cos \frac{p\pi x}{L} \sin \frac{q\pi y}{W} dy dx.$$

According to the orthogonality conditions, each integral on the right is zero, except when $m = p$ and $n = q$. Therefore,

$$\begin{aligned} \int_0^L \int_0^W f(x, y) \cos \frac{p\pi x}{L} \sin \frac{q\pi y}{W} dy dx &= a_{pq} \int_0^L \cos^2 \frac{p\pi x}{L} dx \int_0^W \sin^2 \frac{q\pi y}{W} dy \\ &= \begin{cases} \frac{LW}{4} a_{pq}, & p \neq 0 \\ \frac{LW}{2} a_{pq}, & p = 0. \end{cases} \end{aligned}$$

Hence,

$$a_{pq} = \frac{2}{LW} \int_0^L \int_0^W f(x, y) \sin \frac{q\pi y}{W} dy dx, \quad q = 1, 2, 3, \ldots, \tag{44}$$

and for $p \geq 1, q \geq 1$,

$$a_{pq} = \frac{4}{LW} \int_0^L \int_0^W f(x, y) \cos \frac{p\pi x}{L} \sin \frac{q\pi y}{W} dy dx. \tag{45}$$

Finally, the solution to the initial-boundary value problem (29)-(32) is given by (42), where the coefficients are prescribed by (44) and (45). $\square$

Theorem 1. *Let $u(x, t)$ be a continuously differentiable function that satisfies the heat equation*

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{46}$$

and the boundary conditions

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \tag{47}$$

Then $u(x, t)$ attains its maximum value at $t = 0$, for some x in $[0, L]$, that is,

$$\max_{t \geq 0, 0 \leq x \leq L} u(x, t) = \max_{0 \leq x \leq L} u(x, 0).$$

Theorem 2. *The initial-boundary value problem*

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, t > 0, \tag{48}$$

$$u(0, t) = u(L, t) = 0, \qquad t \geq 0, \tag{49}$$

$$u(x, 0) = f(x), \qquad 0 < x < L, \tag{50}$$

has at most one continuously differentiable solution.