10.4 Fourier Cosine and Sine Series

To solve a partial differential equation, typically we represent a function by a trigonometric series consisting of only sine functions or only cosine functions.

Recall that the Fourier series for an odd function defined on $[-L, L]$ consists entirely of sine terms. Thus we might achieve

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \tag{1}$$

by artificially extending the function $f(x), 0 < x < L$ to the interval $(-L, L)$ in such a way that the extended function is odd. That is,

$$f_o(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0, \end{cases}$$

and extending $f_o(x)$ to all x using $2L$-periodicity. $f_o(x)$ is an extension of $f(x)$ because $f_o(x) = f(x)$ on $(0, L)$. This extension is called the *odd $2L$-periodic extension* of $f(x)$. The resulting Fourier series expansion is called a half-range expansion for $f(x)$ because it represents the function $f(x)$ on $(0, L)$.

Similarly, the *even $2L$-periodic extension* of $f(x)$ as the function

$$f_e(x) = \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L < x < 0, \end{cases}$$

with $f_e(x + 2L) = f_e(x)$.

To illustrate the various extensions, let's consider the function $f(x) = x, 0 < x < \pi$. If we extend $f(x)$ to the interval $(-\pi, \pi)$ using π-periodicity, then the extension f is given by

$$\widetilde{f}(x) = \begin{cases} x, & 0 < x < \pi \\ x + \pi, & -\pi < x < 0, \end{cases}$$

with $\widetilde{f}(x + 2\pi) = \widetilde{f}(x)$. In the previous quiz we saw that the Fourier series for $\widetilde{f}(x)$ is

$$\widetilde{f}(x) \sim \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{1}{k} \sin 2kx,$$

which consists of both odd functions (the sine terms) and even functions (the constant term), because the π-periodic extension $\widetilde{f}(x)$ is neither an even nor an odd function. The odd 2π-periodic extension of $f(x)$ is just $f_o(x) = x, -\pi < x < \pi$, which has the Fourier series expansion

$$f_o(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{2}$$

Because $f_o(x) = f(x)$ on the interval $(0, \pi)$, the expansion in (2) is a half-range expansion for $f(x)$. The even 2π-periodic extension of $f(x)$ is the function $f_e(x) = |x|, -\pi < x < \pi$, which has the Fourier series expansion

$$f_e(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x \tag{3}$$

(see Example 2 in §10.3 lecture notes).

The preceding three extensions, the π-periodic function $\widetilde{f}(x)$, the odd 2π-periodic function $f_o(x)$, and the even 2π-periodic function $f_e(x)$, are natural extensions of $f(x)$. The Fourier series expansions for $f_o(x)$ and $f_e(x)$, given in (2) and (3) equal $f(x)$ on the interval $(0, \pi)$. This motivates the following definitions.

Definition. *Let $f(x)$ be piecewise continuous on the interval $[0, L]$. The Fourier cosine series of $f(x)$ on $[0, L]$ is*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \tag{4}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, \ldots. \tag{5}$$

The Fourier sine series of $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \tag{6}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots. \tag{7}$$

The trigonometric series in (4) is the Fourier series for $f_e(x)$, the even $2L$-periodic extension of $f(x)$. The trigonometric series in (6) is the Fourier series for $f_o(x)$, the odd $2L$-periodic extension of $f(x)$. These are called *half-range expansions* for $f(x)$.

Example 1. *Determine (a) the π-periodic extension $\widetilde{f}$, (b) the odd 2π-periodic extension f_o, and (c) the even 2π-periodic extension f_e, for $f(x) = \pi - x, 0 < x < \pi$.*

Example 2. *Compute the Fourier sine series for $f(x) = \pi - x, 0 < x < \pi$.*

Example 3. *Compute the Fourier cosine series for $f(x) = e^{-x}, 0 < x < 1$.*

A mathematical model for source-less the heat flow in a uniform wire whose ends are kept at constant temperature 0 is the following initial value problem, where $u(x, t)$ is the temperature in the wire at location x and time t:

$$\frac{\partial u}{\partial t}(x, t) = \beta \frac{\partial^2 u}{\partial x^2}(x, t), \qquad 0 < x < L, t > 0 \tag{8}$$

$$u(0, t) = u(L, t) = 0, \qquad t > 0 \tag{9}$$

$$u(x, 0) = f(x), \qquad 0 < x < L. \tag{10}$$

Using the method of separation of variables, we may derive the following solution:

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L}. \tag{11}$$

Example 4. *Find the solution to the heat problem*

$$\frac{\partial u}{\partial t} = 5 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, t > 0$$

$$u(0, t) = u(\pi, t) = 0, \qquad t > 0$$

$$u(x, 0) = x(\pi - x), \qquad 0 < x < \pi.$$