10.3 Fourier Series

A *piecewise continuous* function on $[a, b]$ is continuous at every point in $[a, b]$, except possible for a finite number of points at which the function has jump discontinuity. Such function is necessarily integrable over any finite interval.

A function f is *periodic* of period T if $f(x+T) = f(x)$ for all x in the domain of f. The smallest positive value of T is called the *fundamental period.* For example, both $\sin x$ and $\cos x$ have fundamental period 2π, whereas $\tan x$ has fundamental period π. A constant function is periodic with arbitrary period T.

A function f is even when $f(-x) = f(x)$. An even function is symmetric with respect to the y-axis. A function f is odd when $f(-x) = -f(x)$. An odd function is symmetric with respect to the origin. For example, the functions $1, x^2, x^4, \cos x$ are even, whereas the functions $x, x^3, x^5, \sin x, \tan x$ are odd.

Theorem 1. *If f is an even piecewise continuous function on $[-a, a]$, then*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx. \tag{1}$$

If f is an odd piecewise continuous function on $[-a, a]$, then

$$\int_{-a}^{a} f(x)dx = 0. \tag{2}$$

Example 1. *Using the product-to-sum identities, we obtain the following:*

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0 \tag{3}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \end{cases} \tag{4}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \neq 0, \\ 2L, & m = n = 0. \end{cases} \tag{5}$$

Equations (3)-(5) express an orthogonality condition, satisfied by the set of trigonometric functions $\{1 = \cos 0, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$, where $L = \pi$. If each $f_1, \ldots, f_n$ is periodic with period T, then so is every linear combination $c_1 f_1 + \cdots + c_n f_n$. For example, the sum $7 + 3\cos\pi x - 8\sin\pi x + 4\cos 2\pi x - 6\sin 2\pi x$ has period 2.

If the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\}$$

consisting of $2L$-periodic functions, converges for all x, then the function to which it converges will be periodic with period $2L$. Suppose

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\}, \tag{6}$$

where the a_n's and b_n's are constants. We may determine the coefficients $a_0, a_1, b_1, a_2, b_2, \ldots$ by a method similar to Taylor series. For example, to determine a_0:

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L}\frac{a_0}{2}dx + \sum_{n=1}^{\infty} a_n\int_{-L}^{L}\cos\frac{n\pi x}{L}dx + \sum_{n=1}^{\infty} b_n\int_{-L}^{L}\sin\frac{n\pi x}{L}dx.$$

The signed areas cancel each other out and we obtain

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2} dx = a_0 L,$$

and hence

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)dx.$$

Definition. *Let f be a piecewise continuous function on the interval $[-L, L]$. The* Fourier series *of f is the trigonometric series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\}, \tag{7}$$

where the a_n's and b_n's are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \ldots \tag{8}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots \tag{9}$$

Formulas (8) and (9) are called Euler-Fourier formulas.

Remark. *If f is even, then $f(x) \sin \frac{n\pi x}{L}$ is odd. Therefore*

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx = 0, \quad n = 1, 2, 3, \ldots$$

and hence the Fourier series of f consists only of cosine functions, including $\cos \frac{0\pi x}{L} = \cos 0 = 1$.

Example 2. *Compute the Fourier series for*

$$f(x) = |x|, \quad -\pi < x < \pi.$$

10.3.1 Orthogonal Expansions

Suppose we define an inner product of two functions f and g as

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx,$$

where $w(x) \geq 0$ on $[a, b]$ is called a weight function. Then the square of norm of f is:

$$\|f\|^2 = \int_a^b f^2(x)w(x)dx = \langle f, f \rangle. \tag{10}$$

A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be orthogonal if

$$\langle f_m, f_n \rangle = 0, \quad \text{whenever } m \neq n. \tag{11}$$

For example the set of trigonometric functions $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$ is orthogonal on $[-\pi, \pi]$ with respect to the weight function $w(x) = 1$. A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be orthonormal if

$$\langle f_m, f_n \rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{12}$$

An expansion of a function f in terms of an orthogonal system is called an orthogonal expansion. As always, we may determine the coefficients easily. Suppose $\{f_n\}_{n=1}^{N}$ is an orthogonal basis. Then:

$$f(x) = \frac{\langle f, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 + \cdots + \frac{\langle f, f_N \rangle}{\langle f_N, f_N \rangle} f_N. \tag{13}$$

Fourier series are examples of orthogonal expansions.

10.3.2 Convergence of Fourier Series

Notation:

$$f(x^+) = \lim_{h \to 0^+} f(x+h) \quad \text{and} \quad f(x^-) = \lim_{h \to 0^-} f(x-h).$$

Theorem 2. *If f and f' are piecewise continuous on $[-L, L]$, then for any x in $(-L, L)$,*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\} = \frac{1}{2}[f(x^+) + f(x^-)], \tag{14}$$

where a_n's and b_n's are given by the Euler-Fourier formulas (8) and (9). For $x = \pm L$, the series converges to $\frac{1}{2}[f(-L^+) + f(L^-)]$.

In other words, when f and f' are piecewise continuous on $[-L, L]$, the Fourier series converges to $f(x)$ whenever f is continuous at x and converges to the average of the left- and right-hand limits at points where f is discontinuous.

Example 3. *a) Compute the Fourier series for*

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$$

b) Determine the function to which the Fourier series for $f(x)$ converges.

When f is a $2L$-periodic function that is continuous on $(-\infty, \infty)$ and has a piecewise continuous derivative, its Fourier series not only converges at each point, it converges *uniformly* on $(-\infty, \infty)$. This means that for any $\varepsilon > 0$, the graph of the partial sum

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\}$$

will, for all N sufficiently large, lie in an ε-corridor about the graph of f on $(-\infty, \infty)$.

Theorem 3. *Let f be a continuous function on $(-\infty, \infty)$ and periodic with period $2L$. If f' is piecewise continuous on $[-L, L]$, then the Fourier series for f converges uniformly to f on $[-L, L]$ and hence on any interval. That is, for each $\varepsilon > 0$, there exists an integer N_0 (that depends on ε) such that*

$$\left| f(x) - \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\} \right] \right| < \varepsilon,$$

for all $N > N_0$ and all $x \in (-\infty, \infty)$.

Theorem 4. *Let $f(x)$ be continuous on $(-\infty, \infty)$ and $2L$-periodic. Let $f'(x)$ and $f''(x)$ be piecewise continuous on $[-L, L]$. Then, the Fourier series of $f'(x)$ can be obtained from the Fourier series for $f(x)$ by termwise differentiation. In particular, if*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\},$$

then

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} \left\{ -a_n \sin \frac{n\pi x}{L} + b_n \cos \frac{n\pi x}{L} \right\}.$$

Theorem 5. *Let $f(x)$ be piecewise continuous on $[-L, L]$ with Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\}.$$

Then, for any x in $[-L, L]$,

$$\int_{-L}^{x} f(t)dt = \int_{-L}^{x} \frac{a_0}{2} dt + \sum_{n=1}^{\infty} \int_{-L}^{x} \left\{ a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right\} dt.$$