4.7 Change of Basis

Theorem 15. *Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that*

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}} \tag{1}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}}] \tag{2}$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 15 is called the *change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$*. Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$. Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is square, it must be invertible, by the IMT. Left-multiplying both sides of equation (1) by $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ yields

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} [\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}$$

Thus $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates into $\mathcal{B}$-coordinates. That is,

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \tag{3}$$

If $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_{\mathcal{E}} = \mathbf{b}_1$, and likewise for the other vectors in $\mathcal{B}$. In this case, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in §4.4, namely,

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$$

To change coordinates between two nonstandard bases in $\mathbb{R}^n$, we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

Example 1. *Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases for V, and let $P = [[\mathbf{d}_1]_{\mathcal{A}} \quad [\mathbf{d}_2]_{\mathcal{A}} \quad [\mathbf{d}_3]_{\mathcal{A}}]$. Which of the followind equations is satisfied by P for all $\mathbf{x}$ in V? (i) $[\mathbf{x}]_{\mathcal{A}} = P[\mathbf{x}]_{\mathcal{D}}$ (ii) $[\mathbf{x}]_{\mathcal{D}} = P[\mathbf{x}]_{\mathcal{A}}$.*

Example 2. *Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$. Find the change-of -coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.*

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$