4.6 Rank

If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted by Row$A$. Since the rows of A are identified with the columns of A^T, we could also write ColA^T in place or Row$A$.

Theorem 13. *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.*

Definition. *The* rank *of A is the dimension of the column space of A.*

Theorem 14 (The Rank Theorem)**.** *The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation*

$$\text{rank } A + \dim \text{ Nul } A = n$$

Theorem (IMT (continued))**.** *Let A be an $n \times n$ matrix. Then each of the following is equivalent to the statement that A is an invertible matrix.*

13. *The columns of A form a basis of $\mathbb{R}^n$.*
14. Col $A = \mathbb{R}^n$
15. dim Col $A = n$
16. rank $A = n$
17. Nul $A = \{\mathbf{0}\}$
18. *dim Nul A = 0*

Example 1. *Assume that $A \sim B$, where*

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

Example 2. *If a 6×3 matrix A has rank 3, find dim Nul A, dim Row A, and rank A^T.*

Example 3. *If the null space of a 7×6 matrix A is 5-dimensional, what is dim Col A?*

Example 4. *If A is a 6×4 matrix, what is the smallest possible dimension of Nul A?*