## 4.5 The Dimension of a Vector Space

**Theorem 9.** *If a vector space $V$ has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in $V$ containing more than $n$ vectors must be linearly dependent.*

**Theorem 10.** *If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consist of exactly $n$ vectors.*

**Definition.** *If $V$ is spanned by a finite set, then $V$ is said to be finite-dimensional, and the dimension of $V$, written as $\dim V$, is the number of vectors in a basis for $V$. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero If $V$ is not spanned by a finite set, then $V$ is said to be infinite-dimensional.*

**Theorem 11.** *Let $H$ be a subspace of a finite-dimensional vector space $V$. Any linearly independent set in $H$ can be expanded, if necessary, to a basis for $H$. Also, $H$ is finite-dimensional and*

$$\dim H \leq \dim V$$

**Theorem 12** (The Basis Theorem)**.** *Let $V$ be a $p$-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly $p$ elements in $V$ is automatically a basis for $V$. Any set of exactly $p$ elements that spans $V$ is automatically a basis for $V$.*

The dimension of Nul$A$ is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col$A$ is the number of pivot columns in $A$.

**Example 1.** *For the following set, (a) find a basis, and (b) state the dimension.*

$$\left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

**Example 2.** *Determine the dimensions of* Nul$A$ *and* Col$A$ *for*

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$