4.4 Coordinate Systems

Theorem 7 (The Unique Representation Theorem)**.** *Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x} \in V$, there exists a unique set of scalars $c_1, \dots, c_n$ such that*

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Proof. Since $\mathcal{B}$ spans V, there exists sclars such that (1) holds. Suppose $\mathbf{x}$ also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n$$

for scalars $d_1, \dots, d_n$. Then, subtracting, we have

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_n - d_n)\mathbf{b}_n \tag{2}$$

Since $\mathcal{B}$ is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for all $1 \leq j \leq n$. □

Definition. *Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.*

If $c_1, \dots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the corrdinate vector of $\mathbf{x}$ relative to $\mathcal{B}$, or the $\mathcal{B}$-coordinate vector of $\curvearrowleft$. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping determined by $\mathcal{B}$.

Example 1. *Find the vector $\mathbf{x}$ determined by coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$ and the basis*

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}$$

The standard basis for $\mathbb{R}^n$ is formed by the columns of I_n: $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Thus $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$.

When a basis $\mathcal{B}$ for $\mathbb{R}^n$ is fixed, the $\mathcal{B}$-coordinate vector of a specified $\mathbf{x}$ is easily found.

Example 2. *Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$ relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.*

Solution. The $\mathcal{B}$-coordinates c_1, c_2, c_3 of $\mathbf{x}$ satisfy

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} \tag{3}$$

The augmented matrix from Equation (3) row reduces to

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

So $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$ and $\mathbf{x} = -2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. □

The matrix in (3) changes the $\mathcal{B}$-coordinates of a vector $\mathbf{x}$ into the standard coordinates for $\mathbf{x}$. An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$$

Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \tag{4}$$

We call $P_{\mathcal{B}}$ the change of coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Left multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.

Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by IMT). Left-multiplication by $P_{\mathcal{B}}^{-1}$ converts $\mathbf{x}$ into its $\mathcal{B}$-coordinate vector:

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$, produced here by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping mentioned earlier. Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping is a 1-1 linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$, by IMT.

Theorem 8. *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a 1-1 linear transformation from V onto $\mathbb{R}^n$.*

The linearity of the coordinate mapping extends to linear combinations. If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, c_p$ are scalars, then

$$[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p]_{\mathcal{B}} = c_1[\mathbf{u}_1]_B + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}} \tag{5}$$

In words, (5) says that the $\mathcal{B}$-coordinate vector of a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$ is the *same* linear combination of their coordinate vectors.

The coordinate mapping in Theorem 8 is an important example of an *isomorphism* from V onto $\mathbb{R}^n$. In general, a 1-1 linear transformation from a vector space V onto a vector space W is called an *isomorphism* from V onto W. The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces. Every vector space calculation in V is accurately reproduced in W, and vice versa. In particular, any real vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$.

Example 3. *Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis of the space $\mathbb{P}_3$ of polynomials. A typical element $\mathbf{p} \in \mathbb{P}_3$ has the form*

$$\mathbf{p}(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Since $\mathbf{p}$ is already displayed as a linear combination of the standard basis vectors, we conclude that

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Thus the coordinate mapping $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{B}}$ is an isomorphism from $\mathbb{P}_3$ onto $\mathbb{R}^4$. All vector space operations in $\mathbb{P}_3$ correspond to operations in $\mathbb{R}^4$.