

4.3 Linearly Independent Sets; Bases

Recall that a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be linearly independent if the vector equation

$$c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0} \quad (1)$$

has only the trivial solution $c_1 = \dots = c_p = 0$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent if the equation (1) has a nontrivial solution.

Theorem 4. *An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_i \neq \mathbf{0}$, is linearly dependent if and only if some \mathbf{v}_j is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$, with $j > 1$.*

Definition. *Let H be a subspace of a vector space V . An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a basis for H if*

1. \mathcal{B} is a linearly independent set, and
2. the subspace spanned by \mathcal{B} coincides with H , that is,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

Example 1. *Determine which of the following sets are bases for \mathbb{R}^3 . Of the sets that are not bases, determine which ones are linearly independent and which ones span \mathbb{R}^3 . Justify your answers.*

1. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
2. $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

Theorem 5 (The Spanning Set Theorem). *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V , and let $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.*

1. *If one of the vectors in S , say \mathbf{v}_k , is a linear combination of the remaining vectors in S , then the set formed from S by removing \mathbf{v}_k still spans H .*
2. *If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H .*

Example 2. *Find a basis for the space spanned by the the following vectors.*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

Theorem 6. *The pivot columns of a matrix A form a basis for $\text{Col}A$.*

Example 3. *Assume that A is row equivalent to B . Find bases for $\text{Nul}A$ and $\text{Col}A$.*

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4. *Let $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$. It can be verified that $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.*