4.2 Null Spaces, Column Spaces, and Linear Transformations

Definition. *The* null space *of an $m \times n$ matrix A, written as* Nul$A$*, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,*

$$\text{Nul}A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Example 1. *Determine if $\mathbf{w} = (1, -1, 1)$ in in* Nul$A$*, where* $A = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}$

Theorem 2. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Example 2. *Find an explicit description of* Nul$A$*, by listing vectors that span the null space, for* $A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

Definition. *The* column space *of an $m \times n$ matrix A, written as* Col$A$*, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then*

$$\text{Col}A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

Theorem 3. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Note that

$$\text{Col}A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

Example 3. *For* $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$*, (a) find k such that* Nul$A$ *is a subspace of $\mathbb{R}^k$, and (b) find k such that* Col$A$*. is a subspace of $\mathbb{R}^k$.*

Recall:

Definition. *A linear transformation $T : V \to W$ is a rule that assigns to each vector $\mathbf{x} \in V$ a unique vector $T(\mathbf{x}) \in W$, such that*

1. *$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$, and*
2. *$T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v} \in V$ and all scalars c.*

The *kernel*, or null space of such a T is the set of all $\mathbf{u} \in V$ such that $T(\mathbf{u}) = \mathbf{0}$, the zero vector in W. The *range* of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x} \in V$. If $T\mathbf{x} = A\mathbf{x}$ for the standard matrix A, then the kernel is the null space of A and the range of T is the column space of A. The kernel of T is a subspace of V. Also, the range of T is a subspace of W.

Example 4. *Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [Hint: Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{w})$ for some $\mathbf{x}, \mathbf{w} \in V$.]*

Comparison between Nul$A$ and Col$A$ for an $m \times n$ matrix A:
Nul$A$:

1. Nul$A$ is a subspace of $\mathbb{R}^n$.
2. Nul$A$ is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul$A$ must satisfy.
3. It takes time to find vectors in Nul$A$. Row operations on $[A \quad \mathbf{0}]$ are required.
4. There is no obvious relation between Nul$A$ and the entries in A.
5. A typical vector $\mathbf{v} \in$ Nul$A$ has the property that $A\mathbf{v} = \mathbf{0}$.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul$A$. Just compute $A\mathbf{v}$.
7. Nul$A = \{\nvdash\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
8. Nul$A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

Col$A$:

1. Col$A$ is a subspace of $\mathbb{R}^m$.
2. Col$A$ is explicitly defined; that is, you are told how to build vectors in Col$A$.
3. It is easy to find vectors in Col$A$. The columns of A are displayed; others are formed from them.
4. There an obvious relation between Col$A$ and the entries in A, since each column of A is in Col$A$.
5. A typical vector $\mathbf{v} \in$ Col$A$ has the property that the equation $A\mathbf{v} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col$A$. Row operations on $[A \quad \mathbf{v}]$ are required.
7. Col$A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.
8. Col$A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.