# 4 Vector Spaces

## 4.1 Vector Spaces and Subspaces

**Definition.** *A* vector space *is a nonempty set $V$ of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to ten axioms listed below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and for all scalars $c, d$.*

1. *The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in $V$. This statement means $V$ is closed under vector addition.*
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. *There is a zero vetor $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.*
5. *For each $\mathbf{u} \in V$, there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{u}$.*
6. *The scalar multiple of $\mathbf{u}$ by $c$, denoted by $c\mathbf{u}$, is in $V$. This statment means $V$ is closed under scalar multiplication.*
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

**Example 1.** *The spaces $\mathbb{R}^n$, where $n \geq 1$ is an integer, are premier examples of vector spaces.*

**Definition.** *A* subspace *of a vector space $V$ is a subset $H$ of $V$ that has three properties:*

1. *The zero vector of $V$ is in $H$.*
2. *$H$ is closed under vector addition.*
3. *$H$ is closed under scalar multiplication.*

**Example 2.** *The set consisting of only the zero vector in a vector space $V$ is a subspace of $V$, called the zero subspace and written as $\{\mathbf{0}\}$.*

**Theorem 1.** *If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space $V$, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of $V$.*

We call $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ the subspace spanned, or generated, by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Given any subspace $H$ of $V$, a spanning, or generating, set for $H$ is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $H$ such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

**Example 3.** *Let $W$ be the set of all vectors of the forms shown, where $a, b, c$ represent arbitrary real numbers. In each case, either find a set $S$ of vectors that span $W$ or give an example to show that $W$ is* not *a vector space.*

1. $\begin{bmatrix} 1 \\ 3a - 5b \\ 3b + 2a \end{bmatrix}$
2. $\begin{bmatrix} 4a + 3b \\ 0 \\ a + 3b + c \\ 3b - 2c \end{bmatrix}$