3.3 Cramer's Rule

Theorem 7 (Cramer's Rule)**.** *Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by*

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Theorem 8. *Let A be an invertible $n \times n$ matrix. Then*

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}A$$

where the adjugate, or classical adjoint, of A is the matrix of cofactors:

$$\mathrm{adj}A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem 9. *If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.*

Theorem 10. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then*

$$\{\textit{area of } T(S)\} = |\det A| \cdot \{\textit{area of } S\} \tag{2}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\textit{volume of } T(S)\} = |\det A| \cdot \{\textit{volume of } S\} \tag{3}$$

Example 1. *Use Cramer's rule to compute the solution of*

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ -x_1 + 2x_3 &= 2 \\ 3x_1 + x_2 + 3x_3 &= -2 \end{aligned}$$