3 Determinants

3.1 Introduction to Determinants

Definition. *For $n \geq 2$, the* determinant *of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,*

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n} a_{1n}\det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j}\det A_{1j}$$

A notation for determinant of a matrix is a pair of vertical lines in place of brackets. The (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j}\det A_{ij} \tag{1}$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

This formula is called a cofactor expansion across the first row of A.

Theorem 1. *The determinant of an $n \times n$ matrix A can be computed by cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (1) is*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 1. *Compute the determinant of* $A = \begin{bmatrix} 5 & -2 & 2 \\ 0 & 3 & -3 \\ 2 & -4 & 7 \end{bmatrix}$.

Theorem 2. *If A is a triangulr matrix, then* $\det A$ *is the product of the entries on the main diagonal of A.*