2.3 Characterization of Invertible Matrices

Theorem 8 (The Invertible Matrix Theorem)**.** *Let A be a square $n \times n$ matrix. Then the folllowing are equivalent.*

1. *A is an invertible matrix.*
2. *A is row equivalent to the $n \times n$ identity matrix.*
3. *A has n pivot positions.*
4. *The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.*
7. *The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$.*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*
10. *There is an $n \times n$ matrix C such that $CA = I$.*
11. *There is an $n \times n$ matrix D such that $AD = I$.*
12. *A^T is an invertible matrix.*

The statement 7 in Theorem 8 could also be written as "The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$. The following fact follows from Theorem 8.

Fact. *Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.*

The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: th invertible matrices, and the noninvertible matrices. Each statement in the theorem describes a property of every $n \times n$ invertible matrix. The negation of a statement in the theorem describes a property of every $n \times n$ noninvertible matrix. For instance, an $n \times n$ noninvertible matrix is not row equivalent to I_n, does not have n pivot positions, and has linearly dependent columns.

Recall that matrix multiplication corresponds to composition of linear transformations. When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \tag{1}$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \tag{2}$$

Theorem 9. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. in that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).*

Example 1. *Determine which of the following matrices are invertible.*

1. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$
2. $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$