## 2.2 The Inverse of a Matrix

An $n \times n$ matrix $A$ is said to be *invertible* if there is an $n \times n$ matrix $C$ such that $CA = I$ and $AC = I$, where $I = I_n$, the $n \times n$ identity matrix. In this case, $C$ is an *inverse* of $A$. In fact, $C$ is uniquely determined by $A$, because if $B$ were another inverse of $A$, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

This unique inverse is denoted by $A^{-1}$. Thus

$$A^{-1}A = AA^{-1} = I.$$

**Theorem 4.** *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bd \neq 0$, *then* $A$ *is invertible and*

$$A^{-1} = \frac{1}{ad - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

*If* $ad - bd = 0$, *then* $A$ *is not invertible.*

The quantity $ad - bd$ is called the determinant of $A$ and we write $\det A = ad - bc$. Theorem 4 says that a $2 \times 2$ matrix $A$ is invertible if and only if $\det A \neq 0$.

**Theorem 5.** *If* $A$ *is an invertible* $n \times n$ *matrix, then for each* $\mathbf{b} \in \mathbb{R}^n$, *the equation* $A\mathbf{x} = \mathbf{b}$ *has the unique solution* $\mathbf{x} = A^{-1}\mathbf{b}$.

**Theorem 6.**

1. *If* $A$ *is an invertible matrix, then* $A^{-1}$ *is invertible and* $(A^{-1})^{-1} = A$.
2. *If* $A$ *and* $B$ *are* $n \times n$ *invertible matrices, then so is* $AB$, *and the inverse of* $AB$ *is the product of the inverses of* $A$ *and* $B$ *in the reverse order, that is,* $(AB)^{-1} = B^{-1}A^{-1}$.
3. *If* $A$ *is an invertible matrix, then so it* $A^T$, *and the inverse of* $A^T$ *is the transpose of* $A^{-1}$. *That is,* $(A^T)^{-1} = (A^{-1})^T$.

The generalization of Theorem 6 is that the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

An invertible matrix $A$ is row equivalent to an identity matrix, and we can find $A^{-1}$ by watching the row reduction of $A$ into $I$. An *elementary matrix* is one that is obtained by performing a single elementary row operation on an identity matrix.

**Example 1.** *Let* $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.
*Compute* $E_1A, E_2A, E_3A$ *and describe the elementary row operations on* $A$.

**Fact.** *If an elementary row operation is performed on an* $m \times n$ *matrix* $A$, *the resulting matrix can be written as* $EA$, *where the* $m \times m$ *matrix* $E$ *is created by performing the same row operation on* $I_m$. *Each elementary matrix* $E$ *is invertible. The inverse of* $E$ *is the elementary matrix of the same type that transforms* $E$ *back into* $I$.

**Theorem 7.** *An* $n \times n$ *matrix* $A$ *is invertible if and only if* $A$ *is row equivalent to* $I_n$, *and in this case, any sequence of elementary row operations that reduces* $A$ *to* $I_n$ *also transforms* $I_n$ *into* $A^{-1}$.

Thus to find $A^{-1}$, we row reduce the augmented matrix $[A \quad I]$. If $A$ is row equivalent to $I$, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, $A$ does not have an inverse.

**Example 2.** *Find the inverse of the following matrices, if they exist.*

*1.* $\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$.

*2.* $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$.

Denote the columns of $I_n$ by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then row reduction of $[A \quad I]$ to $[I \quad A^{-1}]$ can be viewed as the simultaneous solution of the $n$ systems

$$A\mathbf{x} = \mathbf{e}_1, \quad \ldots, \quad , A\mathbf{x} = \mathbf{e}_n \tag{1}$$

where the augmented columns of these sysstems have all been replaced next to $A$ fo form $[A\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n] = [A \quad I]$. The equation $AA^{-1} = I$ and the definition of matrix multiplication show that the columns of $A^{-1}$ are precisely the solutions of the systems in (1).