1.9 The Matrix of a Linear Transformation

Theorem 10. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \tag{1}$$

Proof. Write $\mathbf{x} = I_n\mathbf{x} = [\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n]\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, and use the linearity of T to compute

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}$$

□

The matrix A in (1) is called the standard matrix for the linear transformation T.

We know that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation, and vice versa. The term linear transformation focuses on a property of a mapping, while matrix transformation describes how such a mapping is implemented.

Example 1. *Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = e\mathbf{x}$.*

Definition. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$.*

Equivalently, T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$. "Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question.

Definition. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$.*

Equivalently, T is one-to-one if, for each $\mathbf{b} \in \mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all. "Is T one-to-one?" is a uniqueness question.

Theorem 11. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.*

Theorem 12. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:*

1. *T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.*

2. *T is one-to-one if and only if the columns of A are linearly independent.*

Example 2. *Fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.*

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$$