

1.7 Linear Independence

Definition. An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is said to be linearly independent if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be linearly dependent if there exist weights c_1, \dots, c_p , not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0} \quad (1)$$

Equation (1) is called a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ when the weights are not all zero. An indexed set is linearly dependent if and only if it is not linearly independent. We may write the matrix equation $A\mathbf{x} = \mathbf{0}$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_p\mathbf{a}_p = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Fact. A set containing only one vector, say \mathbf{v} , is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$. This is because the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$. The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

Fact. A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Example 1. Determine if the vectors are linearly independent. Justify each answer.

1. $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$

Theorem 7 (Characterization of Linearly Dependent Sets). An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some \mathbf{v}_j , with $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Note that Theorem 7 does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Theorem 8. If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is linearly dependent if $p > n$.

Proof. Let $A = [\mathbf{v}_1 \ \dots \ \mathbf{v}_p]$. Then A is $n \times p$ and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent. \square

Note that Theorem 8 says nothing about the case in which the number of vectors in the set does not exceed the number of entries in each vector.

Theorem 9. If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n contains the zero vector, then the set is linearly dependent.

Proof. By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_p = \mathbf{0}$ shows that S is linearly dependent. \square

Example 2. Determine by inspection whether the vectors are linearly independent. Justify each answer.

1. $\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example 3. Each statement is either true in all cases, or false for at least one example. If false, construct a specific example to show that the statement is not always true. Such an example is called a counterexample to the statement. If a statement is true, give a justification. One specific example cannot explain why a statement is always true.

1. If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in \mathbb{R}^4 and $\mathbf{v}_3 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

2. If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are linearly independent vectors in \mathbb{R}^4 , then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. Hint: Think about $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$.