1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition. *If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,*

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$. The equation $A\mathbf{x} = \mathbf{b}$ is called a matrix equation.

Theorem 3. *If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$, which, in turn, has the same solution set as the system of linear equations whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]$.*

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Theorem 4. *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.*

1. *For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*
2. *Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbb{R}^m$.*
4. *A has a pivot position in every row.*

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Definition. *The matrix with 1's on the diagonal and 0's elsewhere is called an identity matrix and is denoted by I.*

Theorem 5. *If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:*

1. *$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$*
2. *$A(c\mathbf{u}) = c(A\mathbf{u})$*

The proof follows from the definition of matrix multiplication and multiplication in real numbers.

Example 1. *Compute the product using (a) the definition, and (b) the rowvector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.*

1. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$
2. $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Example 2. *Use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.*

1. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

2. $z_1 \begin{bmatrix} 4 \\ -2 \end{bmatrix} + z_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} + z_4 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$

Example 3. *Write the system as a vector equation and then as a matrix equation.*

$$\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$$

Example 4. *Given A and $\mathbf{b}$ in, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Example 5. *Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the following matrix? Do the columns of this matrix span $\mathbb{R}^3$?*

$$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$