1.3 Vector Equations

For this course, unless we explicitly specify, we only work with real numbers. We call a number a *scalar*.

Definition 1. *A* vector *is an ordered list of numbers. A matrix with only one column is called a column vector, or simply a vector. We denote the set of vectors with exactly two real entries by* $\mathbb{R}^2$ *(read "r-two"), where* $\mathbb{R}$ *is the set of real numbers.*

Two vectors are equal if and only if their corresponding entries are equal. Thus $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ are not equal.

There are two operations among vectors and scalars: addition of vectors and multiplication of a vector by a scalar.

We define addition between two vectors $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ as $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$, where a, b, c, d are real numbers.

For a scalar c and a vector $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, we define $c\mathbf{u} = c\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$, where a, b are real numbers. Note that we use boldface letters to indicate vectors and lightface letters to indicate scalars. Sometimes, we may write a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ as (a, b). Note that a column vector is not the same as a row vector, that is, $\begin{bmatrix} a \\ b \end{bmatrix} \neq [a \quad b]$.

We may think of vectors in $\mathbb{R}^2$ as arrows that we may move freely in a two dimensional plane. Each vector has a length and a direction. If we put the initial point of the arrow on the origin, we may denote each vector with an ordered pair that is the coordinate of the terminal point of such an arrow. A geometric description of a vector is that the vector $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose vertices are $\mathbf{u}, \mathbf{v}$, and $\mathbf{0}$, where $\mathbf{0}$ is the zero vector, whose length is zero. The zero vector is the only vector that has no direction.

Similarly, vectors in $\mathbb{R}^3$ are 3×1 column matrices. And $\mathbb{R}^n$ (read "r-n") denotes the collection of all n-tuples, where n is a positive integer. The zero vector has all entries zero.

All vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$, with arbitrary scalars c, d have the following properties.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{u} - \mathbf{u} = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Definition 2. *Given vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ *in* $\mathbb{R}^n$ *and scalars* $c_1, c_2, \ldots, c_p$*, the vector*

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a linear combination *of vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ *with weights* $c_1, c_2, \ldots, c_p$*.*

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix above.

Definition 3. *If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ *are in* $\mathbb{R}^n$*, then the set of all linear combinations of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ *is denoted by* $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *and is called the subset of* $\mathbb{R}^n$ *spanned (or generated) by* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$*. That is,* $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *is the collection of all vectors that can be written in the form*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with scalars $c_1, c_2, \ldots, c_p$*.*

Note that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$, in particular, the zero vector must be in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. The $\text{Span}\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$. Geometrically, $\text{Span}\{\mathbf{v}\}$ is the set of all points on the line in $\mathbb{R}^n$ throught $\mathbf{v}$ and $\mathbf{0}$. Two nonzero vectors in $\mathbb{R}^3$ that are not multiples of each other, define a plane.

Example 1. *Compute* $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{u} - 2\mathbf{v}$*, where* $\mathbf{u} = (3, 2)$ *and* $\mathbf{v} = (2, -1)$*.*

Example 2. *Write a system of equations that is equivalent to the given vector equation.*

$$x_1(-2, 3) + x_2(8, 5) + x_3(1, -6) = (0, 0)$$

Example 3. *Determine if* $\mathbf{b}$ *is a linear combination of* $\mathbf{a}_1, \mathbf{a}_2$*, and* $\mathbf{a}_3$*, where* $\mathbf{a}_1 = (1, -2, 2), \mathbf{a}_2 = (0, 5, 5), \mathbf{a}_3 = (2, 0, 8)$*, and* $\mathbf{b} = (-5, 11, -7)$*.*

Example 4. *List a few vectors in* $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$*, where* $\mathbf{v}_1 = (3, 0, 2)$ *and* $\mathbf{v}_2 = (-2, 0, 3)$*.*