1.2 Row Reduction and Echelon Forms

In the definitions that follow, a nonzero row or column in a matrix means a row or column that contains at least one nonzero entry; a leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

Definition 1. *A rectangular matrix is in* echelon form *(or row echelon form) if it has the following three properties:*

1. *All nonzero rows are above any rows of all zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*
3. *All entries in a column below a leading entry are zeros.*

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form *(or reduced row echelon form):*

4. *The leading entry in each nonzero row is 1.*
5. *Each leading 1 is the only nonzero entry in its column.*

An echelon matrix (respectively, reduced echelon matrix) is one that is in echelon form (respectively, reduced echelon form.)

Theorem 1 (Uniqueness of the Reduced Echelon Form)**.** *Each matrix is row equivalent to one and only one reduced echelon matrix.*

For the proof, we need to wait until we learn about linear independence in vector spaces in chapter 4.

If a matrix A is row equivalent to an echelon matrix U, we call U an echelon form (or row echelon form) of A; if U is in reduced echelon form, we call U the reduced echelon form of A.

Definition 2. *A* pivot *position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A* pivot column *is a column of A that contains a pivot position.*

Example 1. *Row reduce the matrix A below to echelon form, and locate the pivot columns of A.*

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position. Now, interchange rows 1 and 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

There is no way to create a leading entry in column 3! However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns. The pivots in this example are 1, 2 and -5. □

Example 2. *Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:*

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution. STEP 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

STEP 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position. Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 3: Use row replacement operations to create zeros in all positions below the pivot. We could have divided the top row by the pivot, 3, but with two 3s in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 4: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify. With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column. For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below. This produces the following matrix.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row. Steps 1-3 require no work for this submatrix, and we have reached an echelon form of the full matrix. We perform one more step to obtain the reduced echelon form.

STEP 5: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation. The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is the reduced echelon form of the original matrix. □

The combination of steps 1-4 is called the forward phase of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the backward phase.

The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$x_1 - 5x_3 = 1 \quad (1)$$
$$x_2 + x_3 = 4 \quad (2)$$
$$0 = 0 \quad (3)$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called basic variables. The other variable, x_3, is called a free variable. Whenever a system is consistent, as in the above system, the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation. In this last system, solve the first and second equations for x_1 and the second for x_2. (Ignore the third equation; it offers no restriction on the variables.)

$$x_1 = 5x_3 + 1 \quad (4)$$
$$x_2 = -x_3 + 4 \quad (5)$$
$$x_3 \text{ is free} \quad (6)$$

The statement “x_3 is free” means that you are free to choose any value for x_3. Once that is done, the formulas in (4) and (5) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is $(1, 4, 0)$; when $x_3 = 1$, the solution is $(6, 3, 1)$. Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3. The descriptions in (4) and (5) are parametric descriptions of solutions sets in which the free variables act as parameters. Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty. Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. For instance, in system (1), (2), we may add 5 times equation 2 to equation 1 and obtain the following equivalent system

$$x_1 + 5x_2 = 21$$
$$x_2 + x_3 = 4$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set. Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

Theorem 2 (Existence and Uniqueness)**.** *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot columni.e., if and only if an echelon form of the augmented matrix has no row of the form*

$$[0 \dots 0b], \text{ with } b \neq 0.$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least on free variable.

1.2.1 Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.