1 Linear Equations

1.1 Systems of Linear Equations

Definition 1. *A* linear equation *in the variables $x_1, \ldots, x_n$ is an equation in the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex constants.

Example 1. *The equation $x_2 = 2(\sqrt{6} - x_1) + x_3$ is linear, whereas the equation $4x_1 - 5x_2 = x_1x_2$ is not linear, because of the term x_1x_2.*

Definition 2. *A* system of linear equations *is a collection of one or more linear equations. A* solution *of the system is a list of values that makes each equation a true statement when the values are substituted for the variables. The set of all possible solutions is called the* solution set *of the linear system. Linear systems that have the same solution set are* equivalent.

Fact. *A system of linear equations has exactly one of the following:*

1. *no solution*
2. *exactly one solution*
3. *infinitely many solutions*

Definition 3. *A system of linear equations is said to be* consistent *if it has either one solution or infinitely many solutions. A system is* inconsistent *if it has no solution.*

The essential information of a linear system can be recorded compactly in a rectangular array called a *matrix.*

Example 2. *Given the system*

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the coefficient matrix *and*

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

is called the augmented matrix *of the system. An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.*

The *size* of a matrix tells how many rows and columns it has. If m and n are positive integers, an $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. By convention, the number of rows always comes first. For example, the augmented matrix in Example 2 is 3×4 (read "3 by 4").

The basic strategy for solving a linear system is to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.

Example 3. *Solve the system in Example 2.*

Solution.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Thus the solution is $(x_1, x_2, x_3) = (1, 0, -1)$. It is good practice to check the solution. □

There are three elementary row operations on the matrix that result in an equivalent system.

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Definition 4. *Two matrices are called* row equivalent *if there is a sequence of elementary row operations that transforms one matrix into the other.*

Fact. *It is important to note that row operations are reversible.*

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set. With practice, the calculations in row operations become easier and faster. We must learn to perform row operations accurately because we will use them throughout the course.

Two fundamental questions about a linear system are as follows:

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists, is it the only one; that is, is the solution unique?

Example 4. *Determine if the following system is consistent:*

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

We use elementary row operations to arrive at an equivalent triangular form of the system:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation.

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 5/2 \end{aligned}$$

The equation $0 = 5/2$ is a short form of $0x_1 + 0x_2 + 0x_3 = 5/2$. There are no values of x_1, x_2, x_3 that satisfy the system, because the equation $0 = 5/2$ is never true. Thus the original system is inconsistent, that is, has no solution. □