LECTURE NOTES ON PARTIAL DIFFERENTIAL EQUATIONS
MATH 53, UC Berkeley

A **partial differential equation (PDE)** is an equation involving an unknown function u of 2 or more variables and certain of its partial derivatives.

EXAMPLES.

- **Two dimensions.** If $u = u(x, y)$ is a function of two variables, the following expressions are examples of PDE:

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1 \tag{1}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

- **Three dimensions.** If $u = u(x, y, z)$ is a function of three variables, the following expressions are PDE:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0, \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1 \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0. \tag{4}$$

- In these examples, (x, y) represents a point in the plane, and (x, y, z) represents a point in space. Sometimes solutions u of PDE depend also on the variable t that denotes time.

$\square$

The **order** of a partial differential equation is the order of the highest partial derivatives occurring in it.

EXAMPLES. The partial differential equations in (1) and (3) are first-order PDE, and those in (2) and (4) are second-order. $\square$

Many of the fundamental laws of the physical sciences are first- or second-order PDE of various sorts, most of which are extremely difficult to understand. In these notes we will consider some important model PDE and discuss a few of their solutions.

The one-dimensional **wave equation** is the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{5}$$

for the unknown function of 2 variables $u = u(x, t)$, where $c > 0$ is a constant. The physical interpretation is that $u(x, t)$ represents the displacement of a vibrating string at the time t and at the point x along the string.

The constant c is the speed of propagation of disturbances moving along the string. To see this, let us check that if g is any function of a single variable, then

$$u(x, t) = g(x - ct) \tag{6}$$

solves the wave equation (5). Using the chain rule, we compute

$$\frac{\partial u}{\partial t} = -cg'(x - ct), \ \frac{\partial^2 u}{\partial t^2} = c^2 g''(x - ct)$$

and

$$\frac{\partial u}{\partial x} = g'(x - ct), \ \frac{\partial^2 u}{\partial x^2} = g''(x - ct).$$

Therefore u given by (6) does indeed solve the wave equation (5).

Note that since $c > 0$, the expression (6) represents a **traveling wave** solution which moves to the right (without changing the shape of its profile) at speed c. Likewise

$$u(x, t) = h(x + ct) \tag{7}$$

is a left moving traveling wave solution of (5).

It can be shown that every solution of the one-dimensional wave equation has the form

$$u(x, t) = g(x - ct) + h(x + ct)$$

for appropriate functions of one variable g and h. This means that every solution is a superposition of a right and of a left moving traveling wave.

LAPLACE'S EQUATION

Laplace's equation in two dimensions is the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{8}$$

for the unknown function of 2 variables $u = u(x, y)$. There are several physical interpretations of this PDE, one of which is that u represents the equilibrium concentration at the point (x, y) in two dimensions of some physical quantity.

• It is not hard to show that various polynomials in 2 variables solve Laplace's equation, for example

$$u(x, y) = x^2 - y^2, \quad u(x, y) = x^3 - 3xy^2, \quad u(x, y) = x^4 - 6x^2y^2 + y^4.$$

• As a more interesting example, let us check that

$$u(x, y) = \log(x^2 + y^2)$$

solves Laplace's equation (except at the singular point $(0, 0)$). We compute

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2}.$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2},$$

and therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4}{x^2 + y^2} - \frac{4(x^2 + y^2)}{(x^2 + y^2)^2} = 0$$

for $(x, y) \neq (0, 0)$.

• We can show likewise that

$$u(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

solves the three-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{9}$$

for $(x, y, z) \neq (0, 0, 0)$.

NOTATION: It is common in engineering and physics to write Laplace's equation as

$$\nabla^2 u = 0 \quad \text{or} \quad \Delta u = 0.$$

Each of these expressions means the same as (8) (for functions of 2 variables) or (9) (for functions of 3 variables).

THE HEAT EQUATION

The one-dimensional **heat equation** (also known as the **diffusion equation**) is the PDE

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} \tag{10}$$

for the unknown function of 2 variables $u = u(x, t)$, where $\sigma > 0$ is a constant. The physical interpretation is that u represents the temperature at time t at the point x along a one dimensional rod, with heat conductivity σ. (Another interpretation is that u represents the concentration at the point x at time t of some chemical diffusing within a one dimensional medium.)

- Let us show that for $t > 0$ the function

$$u(x,t) = \frac{1}{t^{\frac{1}{2}}} e^{-\frac{x^2}{4t}} \tag{11}$$

solves the heat equation (with $\sigma = 1$, to simplify the calculations).

This is a good exercise using the chain rule. We have

$$\frac{\partial u}{\partial x} = -\frac{x}{2t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}}$$

and then

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^{\frac{5}{2}}} e^{-\frac{x^2}{4t}}.$$

We also compute that

$$\frac{\partial u}{\partial t} = -\frac{1}{2t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^{\frac{5}{2}}} e^{-\frac{x^2}{4t}}.$$

Therefore

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

and so u given by (11) does indeed solve the heat equation (with $\sigma = 1$) where $t > 0$.

Bell shaped curves. If we multiply our solution (11) by an appropriate constant, we find that another solution is

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}.$$

The graph of u at each time $t > 0$ is a bell-shaped curve centered at 0, with area 1 under the curve. This is called a **normal distribution** or **Gaussian distribution**, extremely important in probability and statistics theory.

• Readers may wish to check that similarly

$$u(x,y,t) = \frac{1}{4\pi t} e^{-\frac{x^2+y^2}{4t}} \tag{12}$$

solves the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{13}$$

which models diffusion in two spatial dimensions.

The graph of this function u at each time $t > 0$ is a bell-shaped surface centered at $(0,0)$, with volume 1 under the surface. This is a two dimensional normal distribution.