16.9 The Divergence Theorem

Consider the following: Fundamental Theorem of Calculus:

$$F(b) - F(a) = \int_a^b F'(x)\, dx.$$

Green's Theorem:

$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_D \text{div } \vec{F}(x,y)\, dA$$

where C is the positively oriented boundary curve of the plane region D.

The version of Green's Theorem for vector fields on $\mathbb{R}^3$ would be:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E \text{div } \vec{F}(x,y,z)\, dV$$

where S is the boundary surface of the solid region E. This version is called the *Divergence Theorem.* The boundary of E is a closed surface and we use the convention that the positive orientation is outward, that is, the unit normal vector $\vec{n}$ os directed outward from E.

Notice the similarity between this formula and the Fundamental Theorem of Calculus and Green's Theorem: In all of these formulas, we have the derivative of a function on one side and the function evaluated on the boundary on the other side.

Theorem (The Divergence Theorem)**.** *Let E be a solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then*

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F}(x,y,z)\, dV.$$

In words, the Divergence Theorem states that the flux of $\vec{F}$ across the boundary surface of E is the same as the triple integral of the divergence of $\vec{F}$ over E.

Example 1. *Let $\boldsymbol{F}(x,y,z) = z \tan^{-1}(y^2)\hat{\imath} + z^3 \ln(x^2+1)\hat{\jmath} + z\hat{k}$. Find the flux of $\boldsymbol{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.*

Example 2. *a) Given that $\boldsymbol{F}(x,y) = \langle x, y^2 \rangle$, use definition of divergence to determine whether the points $P_1(-1,1)$ and $P_2(-1,-1)$ are sources or sinks for the vector field $\boldsymbol{F}$.*

b) Use a graph of the vector field $\boldsymbol{F}$ and the location of the points P_1 and P_2 to explain and verify the answer in part (a).

Example 3. *Suppose that E and S satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives. Prove:*

a)

$$V(E) = \frac{1}{3} \iint_S \boldsymbol{F} \cdot d\boldsymbol{S},$$

where $\boldsymbol{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

b)

$$\iint_S D_n f\, dS = \iiint_E \nabla^2 f\, dV$$

c)

$$\iint_S (f\nabla g - g\nabla f) \cdot \boldsymbol{n} dS = \iint_E (f\nabla^2 g - g\nabla^2 f)\, dV$$

Homework

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