16.8 Stoke's Theorem

We saw a higher dimensional version of Green's Theorem in the Divergence Theorem. Stoke's Theorem is another higher dimensional version of Green's Theorem, in a different way. Green's Theorem relates a double integral over a plane region D to a line integral around the boundary curve C of D. Stoke's Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S. Note the difference: S is a space surface with a space curve, whereas C is a plane curve around a plane region D.

A positive orientation of the boundary curve C is when we walk with our head in the direction of the unit normal vector $\hat{n}$, the surface S is always on our left.

Theorem (Stoke's Theorem)**.** *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \vec{F} \cdot d\vec{r} = \iint\limits_S \text{curl } \vec{F} \cdot d\vec{S}.$$

We may regard Green's Theorem as a special case of Stoke's Theorem, when S is a flat region on a plane. In this case, $\hat{n} = \hat{k}$ and Stoke's Theorem becomes:

$$\int_C \vec{F} \cdot d\vec{r} = \iint\limits_S \text{curl } \vec{F} \cdot d\vec{S} = \iint (\text{curl } \vec{F}) \cdot \hat{k} \; dA.$$

This is the version of Green's Theorem that involved a curl.

Remark. *Suppose $\vec{F} = \langle P, Q \rangle$. Furthermore, suppose we traverse the curve C for $a \le t \le b$. Then*

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_a^b \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) \; dt \\ &= \int_a^b \left(\vec{F} \cdot \frac{ds}{dt} \vec{T} \right) \; dt \\ &= \int_C (\vec{F} \cdot \vec{T}) \; ds,\end{aligned}$$

where $\vec{T}$ is the unit tangent vector at each point. In other words, we can regard our original definition of the line integral of a vector field as a special case of the line integral of a scalar field, the scalar in this case being the tangential component of the vector field. Thus we may write our two versions of Green's Theorem:

$$\int_C (\vec{F} \cdot \vec{T}) \; ds = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \; dA \quad (\textit{work done by a force } \vec{F} \textit{ around } C)$$

$$\int_C (\vec{F} \cdot \hat{n}) \; ds = \iint\limits_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \; dA \quad (\textit{rate of fluid flow past } C \textit{ if } \vec{F} \textit{ is the flux}).$$

Note that the physical interpretations of $\vec{F}$ is different in these versions. On the other hand, $\vec{F}$ might be the flux of a fluid in the first version as well; the integral of its tangential component is then called the circulation around C: there is a whirlpool inside C if the integral is nonzero. This is the reason for the name "curl." So the line integral of the tangential component of $\vec{F}$ along curve C is the double integral of the vertical component of curl $\vec{F}$ *over the region D enclosed by C. The line integral of the normal component of $\vec{F}$ is*

$$\int_C (\vec{F} \cdot \hat{n}) \; ds = \iint\limits_D (\nabla \cdot \vec{F}) \; dA = \iint\limits_D \text{div } \vec{F} \; dA = \iint\limits_D (\text{curl } \vec{F}) \cdot \hat{k} \; dA.$$

From the mathematical point of view, none of this matters. Although of course it is important in applications. Both versions of Green's Theorem are correct for any vector field.

Example 1. *Verify Stoke's Theorem is true for the vector field*

$$\vec{F}(x, y, z) = \langle -2yz, y, 3x \rangle$$

and surface S, where S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward.

Example 2. *Let $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ and let C be the boundary of the first-octant part of the plane $x + y + z = 1$, going from $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$ and back to $(1, 0, 0)$. Verify Stoke's Theorem.*

Homework

§16.8, page 1139: 13, 15.