## 16.7 Surface Integrals

### Parametric Surfaces

Suppose we have a function $f(x, y, z)$ over a surface $S$, given by

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle, \quad (u, v) \in D.$$

We divide the region $D$ into small subregions, each with dimensions $\Delta u \times \Delta v$. Then the surface $S$ is also divided into corresponding patches $S_{ij}$. We evaluate the function $f$ at some point $P_{ij}^*$ in each patch and multiply the function value by the area of the patch, $\Delta S_{ij}$. Adding all these products, would give us the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

The limit of this Riemann sum, as the number of patches increases without bound, is the *surface integral of $f$ over the surface $S$*:

$$\iint_S f(x, y, z)\ dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

And

$$\iint_S f(x, y, z)\ dS = \iint_D f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v|\ dA.$$

**Remark.** *We saw that we may integrate a function over a curve, called a line integral. If the function is* 1*, then the line integral is the length of the curve:*

$$\int_C f(x, y, z)\ ds = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|\ dt$$

*and*

$$\int_C 1\ ds = L.$$

*Similarly, the surface integral is the integral over a surface. If the function is* 1*, the surface integral gives us the area of the surface.*

$$\iint_S 1\ dS = \iint_D |\vec{r}_u \times \vec{r}_v|\ dA = A(S)$$

*as we saw last time.*

When $z = g(x, y)$, we can regard $x$ and $y$ as the parameters. Then the surface integral becomes

$$\iint_S f(x, y, z)\ dS = \iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA.$$

As usual, if $S$ is a surface with equation $y = h(x, z)$, then we would regard $x$ and $z$ as the parameters.

**Example 1.** *Evaluate*

$$\iint_S y^2 z^2\ dS,$$

*where $S$ is the part of the cone $y = \sqrt{x^2 + z^2}$ given by $0 \le y \le 5$.*

**Oriented Surfaces**

**Example 2.** *The surface with parametric equations*

$$x = 2\cos\theta + r\cos(\theta/2), \quad y = 2\sin\theta + r\cos(\theta/2), \quad z = r\sin(\theta/2), \quad -\frac{1}{2} \le r \le \frac{1}{2}, 0 \le \theta \le 2\pi$$

*is called a* Möbius strip. *A Möbius strip has only one side. We say a Möbius strip is a nonorientable surface.*

From now on, we only consider orientable surfaces, that is, surfaces that have two sides.

We begin with a surface that has a tangent plane at every point on $S$. There are two possible choices for a unit normal vector at each point, call them $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$. If it is possible to choose a unit normal vector $\vec{n}$ at every point $(x, y, z)$ so that $\vec{n}$ varies continuously over $S$, we say $S$ is an *oriented surface* and the given choice of $\vec{n}$ provides $S$ with an *orientation.* There are two possible orientations for any orientable surface. For a closed surface, that is, a surface that is the boundary of a solid region $E$, the convention is that the positive orientation is the one for which the normal vectors point outward from $E$. The inward-pointing normals give the negative orientation.

**Surface Integrals of Vector Fields**

Suppose $S$ is an oriented surface with unit normal vector $\vec{n}$. Suppose $S$ is porous, like a fishing net across a stream, and the stream flowing through $S$ with density $\rho(x, y, z)$ and velocity field $\vec{v}(x, y, z)$. The rate of flow, mass per unit time per unit area, is $\rho\vec{v}$. If we divide $S$ into small patches, the mass of the stream per unit time crossing a small patch $S_{ij}$ in the direction of $\vec{n}$ is approximately $(\rho\vec{v} \cdot \vec{n})A(S_{ij})$, where $\rho, \vec{v}$, and $\vec{n}$ are evaluated at some point on $S_{ij}$. We may add all these quantities and obtain the following integral as the result:

$$\iint_S \rho\vec{v} \cdot \vec{n}\; d\vec{S}.$$

The above integral is the rate of flow through $S$. If $\vec{F} = \rho\vec{v}$, then the integral becomes

$$\iint_S \vec{F} \cdot \vec{n}\; dS.$$

**Definition 1.** *If $\vec{F}$ is a continuous vector field on an oriented surface $S$ with unit normal vector $\vec{n}$, then* surface integral *of $\vec{F}$ over $S$ is*

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n}\; dS.$$

*This integral is also called the* flux *of $\vec{F}$ across $S$.*

The above formula means the surface integral of a vector field over $S$ is equal to the surface integral of its normal component over $S$.

If $S$ is given by $\vec{r}(u, v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

and

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\; dS \\ &= \iint_D \left[\vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\right] |\vec{r}_u \times \vec{r}_v|\; dA.\end{aligned}$$

Therefore,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \; dA.$$

If we have $z = g(x, y)$, then we may take $x$ and $y$ as the parameters, and

$$\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = \langle P, Q, R \rangle \cdot \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle .$$

Thus the formula for flux becomes

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left( -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) \; dA.$$

The above formula is for when $S$ is upward-oriented. For a downward orientation, we multiply by $-1$. Similar formulas apply when $y = h(x, z)$ or $x = k(y, z)$.

**Example 3.** *Evaluate the surface integral* $\iint_S \vec{F} \cdot d\vec{S}$ *for the vector field*

$$\vec{F}(x, y, z) = x \; \hat{\imath} + y \; \hat{\jmath} + 5 \; \hat{k}$$

*and the oriented surface $S$, where $S$ is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$.*

The flux is not just for a fluid. If $\vec{E}$ is an electric field, then the surface integral $\iint_S \vec{E} \cdot d\vec{S}$ is the electric flux of $\vec{E}$. Gauss's Law says that the net charge enclosed by a closed surface $S$ is

$$Q = \varepsilon_0 \iint_S \vec{E} \cdot d\vec{S},$$

where $\varepsilon_0$ is a constant, called the permitivity of free space.

**Homework**

§16.7, page 1132: 16, 18, 23, 24, 26, 29.