16.4 Green's Theorem

Review

For a simply connected region D and a simple curve C on D, we have:

$$\int_C \vec{F} \cdot d\vec{r} = 0 \text{ for every close path } C \Leftrightarrow \int_C \vec{F} \cdot d\vec{r} \text{ is independent of path } \Leftrightarrow \vec{F} \text{ is conservative } (\vec{F} = \nabla f).$$

For Green's Theorem, we use a positively oriented curve. A *positive orientation* of a simple closed curve C refers to a single counterclockwise traversal of C. Thus the region D enclosed by C is always on the left as the point $\vec{r}(t)$ traverses C. Negative orientation is the reverse direction.

Theorem (Green's Theorem)**.** *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then*

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_C P \, dx + Q \, dy.$$

Compare Green's Theorem with the Fundamental Theorem of Calculus:

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

In both cases we have an integral of derivatives on the left side ($F', \partial Q/\partial x$, and $\partial P/\partial y$). And in both cases, on the right side, we have the original functions (F, Q, and P) on the *boundary* of the domain: in the one-dimensional case, the boundary is just the two points a and b from the interval $[a, b]$; in the line integral, the boundary of D is the curve C.

There are other notations for the line integral:

$$\oint_C P \, dx + Q \, dy = \circlearrowleft\!\!\!\!\oint_C P \, dx + Q \, dy = \int_{\partial D} P \, dx + Q \, dy.$$

Example 1. *Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem: $\oint_C y \, dx - x \, dy$, where C is the circle with center the origin and radius 4.*

We may use Green's Theorem to compute area of D, which is $A = \iint_D 1 \, dA$. Thus we would like to have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

We may get the above result with many combinations of P and Q. For example,

$$P = 0, Q = x \Rightarrow A = \oint_C x \, dy$$

$$P = -y, Q = 0 \Rightarrow A = -\oint_C y \, dx$$

$$P = -\frac{1}{2}y, Q = \frac{1}{2}x \Rightarrow A = \frac{1}{2}\oint_C x \, dy - y \, dx.$$

Example 2. *Use Green's Theorem to evaluate the line integral $\int_C y^4 \, dx + 2xy^3 \, dy$ along the positively oriented curve C, where C is the ellipse $x^2 + 2y^2 = 2$.*

Example 3. *Use Green's Theorem to evaluate $\int_C \langle \sqrt{x^2+1}, \tan^{-1} x \rangle \cdot d\vec{r}$, where C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$.*

Example 4. *A particle starts at the origin, moves along the x-axis to $(5,0)$, then along the quarter-circle $x^2 + y^2 = 25, x \geq 0, y \geq 0$ to the point $(0,5)$, and then down the y-axis back to the origin. Use Green's Theorem to find the work done on this particle by the force field $\vec{F}(x,y) = \langle \sin x, \sin y + xy^2 + \frac{1}{5}x^3 \rangle$.*

Example 5. *Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.*

Homework

§16.4: 3, 9, 13, 19, 25.