16.3 The Fundamental Theorem for Line Integrals

Recall that by the Fundamental Theorem of Calculus we have

$$\int_a^b F'(x)\ dx = F(b) - F(a),$$

where F' is continuous on $[a, b]$. Thus the integral of a rate of change is the net change.

Now, if we think of ∇f as a derivative of f, we can see the following version of the Fundamental Theorem for line integrals.

Remark. *The following theorem says that we can evaluate the line integral of a conservative vector field, that is, the gradient field of a potential function f, simply by knowing the value of f at the endpoints of C. Therefore, the line integral of a conservative vector field is independent of the path C, it only depends on the initial point and the terminal point of C.*

Theorem. *Let C be a smooth curve given by the vector function $\vec{r}(t), a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Proof.

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r'}(t)\ dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)\ dt \\ &= \int_a^b \frac{d}{dt} f(\vec{r}(t))\ dt \\ &= f(\vec{r}(b)) - f(\vec{r}(a)).\end{aligned}$$

□

A curve is called *closed* if the terminal point and the initial point are the same, that is, if $\vec{r}(a) = \vec{r}(b)$. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path, then the line integral over a closed path is zero.

Theorem. *$\int_C \vec{F} \cdot d\vec{r}$ is independent of path if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C.*

Proof. ($\Rightarrow$) Suppose $\int_C \vec{F} \cdot d\vec{r}$ is independent of path. We want to show that $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C.

Let C be an arbitrary closed path containing distinct points A and B. Let C_1 and C_2 be the two distinct paths from A to B along C. Then we have $C = C_1 + (-C_2)$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0,$$

because by the assumption $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$.

($\Leftarrow$) Suppose $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C. We want to show that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

Let C_1 and C_2 be arbitrary distinct paths from point A to point B. Then $C = C_1 + (-C_2)$ is a closed path and

$$0 = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}.$$

Therefore $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$. □

Theorem. *Suppose that $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D. That is, there exists a function f such that $\nabla f = \vec{F}$.*

The question is how to determine whether or not a vector field $\vec{F}$ is conservative? Suppose that we know $\vec{F} = \langle P, Q\rangle$ is a conservative, where P and Q have continuous first-order partial derivatives. Then we know that there exists a function f such that $\vec{F} = \nabla f$ and

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}.$$

Then, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

Theorem. *If $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

A simple curve is a curve that does not intersect itself. A simply connected region is a connected region D such that every simple closed curve in D encloses only points that are in D. Thus a simply connected region contains no hole and cannot consist of two separate pieces. The converse of the above theorem is true for these regions and curves.

Theorem. *Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ be a vector field on an open simply connected region D. Suppose that P and Q have continuous first-order partial derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \textit{throughout } D.$$

Then $\vec{F}$ is conservative.

We use "partial integration" to find the potential function f.

Example 1. *Determine whether or not $\vec{F}$ is a conservative vector field. If it is, find a function f such that $\vec{F} = \nabla f$.*

$$\vec{F}(x, y) = ye^x\hat{\imath} + (e^x + e^y)\hat{\jmath}.$$

Example 2. *For*

$$\vec{F}(x, y) = (1 + xy)e^{xy}\hat{\imath} + x^2e^{xy}\hat{\jmath}$$
$$C : \vec{r}(t) = \cos t\hat{\imath} + 2\sin t\hat{\jmath}, \quad 0 \le t \le \frac{\pi}{2},$$

(a) find a function f such that $\vec{F} = \nabla f$

(b) use part (a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve C.

Example 3. *Show that the line integral is independent of path and evaluate the integral.*

$$\int_C \sin y \, dx + (x \cos y - \sin y) \, dy,$$

C is any path from $(2, 0)$ to $(1, \pi)$.