16.2 Line Integrals

Recall that we defined the single integral over an interval $[a, b]$: $\int_a^b f(x)\, dx$. Now we would like to define the single integral over a *curve* C, not just the interval on the x-axis. Suppose C is a curve in $\mathbb{R}^2$ such that for the position (x, y), each x and y is a function of a parameter t, with $a \leq t \leq b$. In other words, we have $\vec{r}(t) = \langle x(t), y(t)\rangle$. Suppose C is a smooth curve, that is, $\vec{r'}$ is continuous and $\vec{r'}(t) \neq 0$. If we divide the curve C into small segments, or sub-arcs, each with length Δs_i, we define the *line integral of* f *along* C as

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i,$$

if the limit exists.

Recall that the length of curve C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

Then the line integral is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

Effectively, we have

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

The value of the line integral does not depend on the parameterization of the curve, as long as we traverse the curve exactly once as t increases from a to b. Thus our original single integral over an interval becomes a special case, where we evaluate a line integral from $(a, 0)$ to $(b, 0)$, with $x = t$ and $y = 0$:

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx.$$

Just as we could interpret the single integral as the area under a nonnegative curve, we may interpret the line integral of a nonnegative curve as the area of the "curtain" under the curve.

Example 1. *Evaluate the line integral*

$$\int_C \frac{x}{y}\, ds, C : x = t^3, y = t^4, 1 \leq t \leq 2.$$

If C is a piecewise smooth curve, that is, if C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where each C_i is a smooth curve and the initial point of C_{i+1} is the terminal point of C_i, then we define the line integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds.$$

We may express a line integral in terms of x or in terms of y:

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))x'(t)\, dt, \qquad \text{line integral with respect to } x$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))y'(t)\, dt, \qquad \text{line integral with respect to } y.$$

Thus our original line integral is a line integral with respect to arc length. It is customary to write the line integrals with respect to x and y together:

$$\int_C P(x,y)\ dx + \int_C Q(x,y)\ dy = \int_C P(x,y)\ dx + Q(x,y)\ dy.$$

We may have to set up the parametric equations so that we would start at a point on C and end at another point on C. Recall the parametric equation for a line segment that starts at $P_0(x_0, y_0)$ and ends at $P_1(x_1, y_1)$:

$$\vec{r}(t) = \langle x_0, y_0\rangle + t\langle x_1 - x_0, y_1 - y_0\rangle, \quad 0 \le t \le 1.$$

In general, the value of the line integral depends on the path as well as on the endpoints of the curve. Also, for line integrals with respect to x or with respect to y, if we switch the direction, the sign of the integral also reverses.

$$\int_{-C} f(x,y)\ dx = -\int_C f(x,y)\ dx \quad \text{and} \quad \int_{-C} f(x,y)\ dy = -\int_C f(x,y)\ dy.$$

This is because Δx_i and Δy_i change sign when we reverse the orientation of C. However, since Δs_i is always positive, the sign of the line integral with respect to arc length does not change when we reverse the traverse of the path:

$$\int_{-C} f(x,y)\ ds = \int_C f(x,y)\ ds.$$

We may apply the same concepts for line integrals in space $\mathbb{R}^3$. Thus

$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\ dt = \int_a^b f(\vec{r}(t))|\vec{r'}(t)|\ dt,$$

where $a \le t \le b$. And when $f(x,y,z) = 1$, we would get the arc length:

$$\int_C\ ds = \int_a^b |\vec{r'}(t)|\ dt = L.$$

Line Integrals of Vector Fields

Recall that the work done by a variable force $f(x)$ in moving an object from a to b along the x-axis is $W = \int_a^b f(x)\ dx$. The work done by a constant force $\vec{F}$ in moving a particle from a point P to another point Q in space is $W = \vec{F}\cdot\vec{PQ}$, where $\vec{PQ}$ is the displacement vector. If $\vec{T}(t_i)$ is the unit tangent vector at point P_i, then the work done by the force field $\vec{F}$ is the limit of the Riemann sum

$$\sum_{i=1}^{n}[\vec{F}(x_i,y_i,z_i)\cdot\vec{T}(x_i,y_i,z_i)]\Delta s_i,$$

that is,

$$W = \int_C \vec{F}(x,y,z)\cdot\vec{T}(x,y,z)\ ds = \int_C \vec{F}\cdot\vec{T}\ ds.$$

Definition 1. *Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by vector function $\vec{r}(t), a \le t \le b$. Then the* line integral of $\vec{F}$ along C *is*

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r'}(t)\ dt = \int_C \vec{F}\cdot\vec{T}\ ds,$$

where $\vec{T} = \vec{r'}(t)/|\vec{r'}(t)|$ is the unit tangent vector at the point (x,y,z). Therefore $d\vec{r} = \vec{r'}(t)\ dt$.

Suppose that $\vec{F} = \langle P, Q, R \rangle$. Using the above definition, we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r'}(t)\ dt \\ &= \int_a^b \langle P, Q, R \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle\ dt \\ &= \int_a^b [Px' + Qy' + Rz']\ dt.\end{aligned}$$

The last integral is precisely the line integral. Therefore,

$$\int_C \vec{F} \cdot\ d\vec{r} = \int_C P\ dx + Q\ dy + R\ dz,$$

where $\vec{F} = \langle P, Q, R \rangle$.

Example 2. *Evaluate* $\int_C \vec{F} \cdot d\vec{r}$*, where* C *is given by the vector function* $\vec{r}(t)$*.*

$$\begin{aligned}\vec{F}(x, y, z) &= \langle x, y, xy \rangle \\ \vec{r}(t) &= \cos t\hat{\imath} + \sin t\hat{\jmath} + t\hat{k}, \quad 0 \leq t \leq \pi.\end{aligned}$$

Remark. *When we reverse the direction of travel along* C*, the unit tangent vector* $\vec{T}$ *also changes direction. Therefore,*

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}.$$

Example 3. *A* 160*-lb man carries a* 25*-lb can of paint up a helical staircase that encircles a silo with a radius of* 20 *ft. The silo is* 90 *ft high and the man makes exactly three complete revolutions climbing to the top. Suppose there is a hole in the can of paint and* 9 *lb of paint leaks steadily out of the can during the man's ascent. How much work is done?*

Homework

§16.2, page 1084: 1, 3, 17, 20, 32, 42.