# 16 Vector Calculus

## 16.1 Vector Fields

**Definition 1.** *Let $D$ be a set in $\mathbb{R}^2$. A* vector field on $\mathbb{R}^2$ *is a function $\vec{F}$ that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\vec{F}(x, y)$.*

The best way to picture a vector field is to draw arrows at representative points. We may write $\vec{F}$ in terms its *component functions* $P$ and $Q$:

$$\vec{F}(x, y) = \vec{F}(\vec{x}) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath} = \langle P, Q\rangle.$$

**Remark.** *$P$ and $Q$ are scalar functions of $x$ and $y$ and are called* scalar fields*.*

**Definition 2.** *Let $E$ be a subset of $\mathbb{R}^3$. A* vector field on $\mathbb{R}^3$ *is a function $\vec{F}$ that assigns to each point $(x, y, z)$ in $E$ a three-dimensional vector $\vec{F}(x, y, z)$.*

**Example 1.** *Examples of vector fields are ocean currents and air velocity vector fields.*

**Example 2.** *By Newton's Law of Gravitation, we may represent the gravitational force field by*

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x} = \left\langle \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle.$$

*The force exerted by an electric charge at $Q$ on a charge $q$ located at $(x, y, z)$ with position vector $\vec{x} = \langle x, y, z\rangle$ is*

$$\vec{F}(\vec{x}) = \frac{\varepsilon qQ}{|\vec{x}|^3}\vec{x}.$$

*The electric field of this electric charge $Q$ on $\mathbb{R}^3$ is*

$$\vec{E}(\vec{x}) = \frac{1}{q}\vec{F}(\vec{x}) = \frac{\varepsilon Q}{|\vec{x}|^3}\vec{x}.$$

### Gradient Fields

Recall that the gradient of a function $f(x, y)$ is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$. Therefore $\nabla f$ is really a vector field on $\mathbb{R}^2$ and is called a *gradient vector field.* Similarly for a function $f(x, y, z)$, the gradient field is

$$\nabla f(x, y, z) = f_x(x, y, z)\hat{\imath} + f_y(x, y, z)\hat{\jmath} + f_z(x, y, z)\hat{k}.$$

Recall that that the gradient field is perpendicular to level curves. Also, the gradient vectors are longer where the level curves are closer to each other for steeper terrain, and the gradient vectors are shorter where the level curves are farther apart for less steep terrain.

**Example 3.**

A vector field $\vec{F}$ s called a *conservative vector field* if it is the gradient of some scalar function, that is, if there exists a function $f$ such that $\vec{F} = \nabla f$. Then $f$ is called a *potential function* of $\vec{F}$.

**Example 4.**

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}.$$

*is conservative because its potential function is*

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}.$$

**Example 5.** *Find the gradient vector field of $f(x, y, z) = x^2 y e^{y/z}$.*

### Homework

§16.1, page 1073: 11, 13, 18, 23, 31