15.9 Change of Variables in Multiple Integrals

Once again, we start with the single variable integral. Recall that we may write

$$\int_a^b f(x)\ dx = \int_c^d f(x(t))x'(t)\ dt = \int_c^d f(x(t))\ \frac{dx}{dt}\ dt \tag{1}$$

where $x(t)$ is a function of t, $a = x(c)$, and $b = x(d)$.

Now let's consider functions of two variables. We have already done change of variables when we integrated a function in Cartesian and in polar coordinates.

$$\iint\limits_R f(x,y)\ dA = \iint\limits_S f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

The polar-Cartesian change of variables is a special case of the more general transformation T, from the uv-plane to the xy-plane. A transformation is a function that maps a region S in uv-plane to the region R in the xy-plane. Thus R is the image of S under T. We say the image of (u,v) under T is $(x,y) = T(u,v)$. If T is one-to-one, then T has an inverse map T^{-1} that maps (x,y) to (u,v), that is $T^{-1}(x,y) = (u,v)$. We also assume that T is a C^1 transformation, which means that the functions $x(u,v)$ and $y(u,v)$ have continuous first-order partial derivatives.

Consider a small rectangular region S in the uv-plane with lower-right corner (u_0, v_0) and the sides of lengths Δu and Δv. The position vector of the image of the point (u,v) is $\vec{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath}$. The image of the lower side of S, which is

$$u_0 \le u \le u_0 + \Delta u, v = v_0$$

is $\vec{r}(u, v_0)$. The tangent vector at $(x_0, y_0) = T(u_0, v_0)$ to this image curve is

$$\vec{r}_u = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath}.$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S,

$$u = u_0, v_0, \le v \le v_0 + \Delta v$$

is

$$\vec{r}_v = \frac{\partial x}{\partial v}\hat{\imath} + \frac{\partial y}{\partial v}\hat{\jmath}.$$

When S is small, the region R is also small and we may say R is approximately a parallelogram determined by the vectors

$$\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)$$

and

$$\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0).$$

We know

$$\vec{r}_u = \lim_{\Delta u \to 0} \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u}.$$

Therefore

$$\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u \vec{r}_u.$$

Similarly,

$$\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \approx \Delta v \vec{r}_v.$$

So we are saying that R is approximately a parallelogram with sides $\Delta u\vec{r}_u$ and $\Delta v\vec{r}_v$. The area of R is thus approximately the area of this parallelogram, which is the size of the cross product of the vectors of its sides:

$$|(\Delta u\vec{r}_u) \times (\Delta v\vec{r}_v)| = |\vec{r}_u \times \vec{r}_v|\Delta u\Delta v. \tag{2}$$

The cross product is

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k} \tag{3}$$

The determinant in Equation (3) is called the Jacobian of the transformation and has a special notation.

Definition 1. *The Jacobian of the transformation T given by $x(u,v)$ and $y(u,v)$ is*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Thus the area of R is approximately

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta v \tag{4}$$

where we evaluate the Jacobian at (u_0, v_0).

Now we may divide every region S in uv-plane into small rectangles whose images in xy-plane are small parallelograms. We may write

$$\begin{aligned} \iint\limits_R f(x,y)\ dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A \\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x(u_i,v_i), y(u_j,v_j))\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta v \end{aligned}$$

Thus we have the following theorem.

Theorem 1 (Change of Variables in a Double Integral)**.** *Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then*

$$\iint\limits_R f(x,y)\ dA = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv.$$

The above theorem says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv.$$

Notice the similarity between this theorem and the one-dimensional formula in Equation (1).

Example 1. *Show the formula for integration in polar coordinates using the Jacobian.*

Proof. We have

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then the Jacobian of T is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus the theorem gives

$$\begin{aligned} \iint\limits_R f(x,y)\ dx\ dy &= \iint\limits_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|\ dr\ d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\ dr\ d\theta, \end{aligned}$$

which the same formula that we had before. □

Example 2. *Evaluate the integral*

$$\iint\limits_R (4x+8y)\ dA,$$

where R is the parallelogram with vertices $(-1,3),(1,-3),(3,-1),(1,5)$ and $x = \frac{1}{4}(u+v), y = \frac{1}{4}(v-3u)$.

Example 3. *Evaluate the integral by making an appropriate change of variables.*

$$\iint\limits_R \sin(9x^2+4y^2)\ dA$$

where R is the region in the first quadrant bounded by the ellipse $9x^2+4y^2=1$.

Next time we will see triple integrals. For now, just note that the Jacobian for a transformation T that maps a region S in uvw-space onto a region R in xyz-space is a 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$