15.8 Triple Integrals in Spherical Coordinates

A coordinate system that simplifies the evaluation of triple integrals over regions bounded by spheres or cones, or when there is symmetry about origin, is called the *spherical coordinate system.* The address of each point in spherical coordinate system is of the form (ρ, θ, ϕ), where $\rho \geq 0$ is the distance from the origin to the point, $0 \leq \phi \leq \pi$ is the angle between the positive z-axis and the line connecting origin to the point, and θ is the same as in cylindrical coordinate system.

Example 1. *The equation of the sphere with center at the origin and radius c is $\rho = c$. This simple equation is the reason for naming the system spherical.*

Example 2. *The graph of $\theta = c$ is a vertical half-plane. The graph of $\phi = c$ is a cone with the z-axis as its axis.*

The relationship between (x, y, z) and (ρ, θ, ϕ) is as follows: Since $r = \rho \sin\phi$, we have

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

and by Pythagorean Theorem and distance formula, we have

$$\rho^2 = x^2 + y^2 + z^2.$$

Example 3. *Sketch the solid described by the following inequalities:*

$$1 \leq \rho \leq 2, \frac{\pi}{2} \leq \phi \leq \pi.$$

Example 4. *(a) Find inequalities that describe a hollow ball with diameter 30 cm and thickness 0.5 cm. Explain how you have positioned the coordinate system that you have chosen.*

(b) Suppose the ball is cut in half. Write inequalities that describe one of the halves.

In spherical coordinates, we use a spherical wedge as the small unit for volume:

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

where $a \geq 0, \beta - \alpha \leq 2\pi$, and $d - c \leq 2\pi$. The spherical wedge is approximately a rectangular box with dimensions $\Delta\rho, \rho_i\Delta\phi$, and $\rho_i \sin\phi_k \Delta\theta$. Thus $\Delta V \approx (\Delta\rho)(\rho_i\Delta\phi)(\rho_i \sin\phi_k\Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$. Then the triple integral is the limit of tripe Riemann sum:

$$\iiint\limits_E f(x, y, z)\ dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V.$$

Therefore the formula for triple integrals in spherical coordinates is

$$\iiint\limits_E f(x, y, z)\ dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi,$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}.$$

Thus

$$dV = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi.$$

Example 5. *Use the Jacobian of a transformation that maps a region in $\rho\theta\phi$-space to a region in xyz-space to derive the formula for triple integration in spherical coordinates.*

Example 6. *Page 1050, question 20.*

Example 7. *Evaluate $\iiint\limits_E y^2\ dV$, where E is the solid hemisphere $x^2 + y^2 + z^2 \leq 9, y \geq 0$.*

Example 8. *Find the volume of a sphere of radius a.*