15.7 Triple Integrals in Cylindrical Coordinates

Recall that we may use Cartesian or polar coordinates to address each point on $\mathbb{R}^2$. In $\mathbb{R}^3$, we may use the axes x, y, and z, or we may use the z-axis together with the polar coordinates for the xy-plane. This coordinate system is called *cylindrical.* Thus in cylindrical coordinate system, the address of each point in space is of the form (r, θ, z), where r and θ are polar coordinates of the projection of the point on the plane $z = 0$, and z is the directed distance from the plane $z = 0$. The conversion formulas are the same we had between Cartesian and polar coordinates, with the added z coordinate.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

and

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z.$$

Example 1. *The equation of the cylinder $x^2+y^2 = c^2$ in the space, converted to cylindrical system, is $r = c$. The simplicity of the equation of the cylinder is the reason for the name of the system.*

When the function $f(x, y, z)$ involves the expression $x^2 + y^2$, or when a problem has symmetry around an axis (that we call the z-axis), it is usually useful to convert to cylindrical coordinates in tripe integrals. Consider the following triple integral:

$$\iiint_E f(x, y, z)\ dV.$$

Suppose that

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}.$$

Thus

$$\iiint_E f(x, y, z)\ dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz\right]\ dA.$$

We may convert the outside double integral into polar coordinates. Therefore,

$$\iiint_E f(x, y, z)\ dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(x, y, z)\ r\ dz\ dr\ d\theta.$$

Example 2. *Evaluate $\iiint_E (x - y)\ dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane, and below the plane $z = y + 4$.*

Example 3. *Evaluate*

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz\ dz\ dx\ dy.$$

Example 4. *Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.*