15.6 Triple Integrals

We define triple integrals for functions of three variables. We begin with a function f defined on a box B, where

$$B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}.$$

We divide B into small boxes with equal dimensions $\Delta x \times \Delta y \times \Delta z$. The volume of each small box is $\Delta V = \Delta x \Delta y \Delta z$. The triple integral of f over the box B is the limit of the triple Riemann sum, if the limit exists:

$$\iiint_B f(x, y, z)\ dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}).$$

The triple integrals always exists when f is continuous.

Theorem (Fubini's Theorem for Triple Integrals)**.** *If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then*

$$\iiint_B f(x, y, z)\ dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\ dx\ dy\ dz.$$

Note that there are six possible orders of integration with three variables. We define the triple integral over a general bounded region E by defining a function F such that F agrees with f on E and is zero for points in a box B that encloses E and outside of E. Thus

$$\iiint_E f(x, y, z)\ dV = \iiint_B F(x, y, z)\ dV.$$

We say a region is of type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}.$$

For a type 1 region, we have

$$\iiint_E f(x, y, z)\ dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz \right] dA,$$

where we integrate first with respect to z while we hold x and y fixed, that is, we treat $u_1(x, y)$ and $u_2(x, y)$ as constants.

If the projection D of E onto the xy-plane is a type I plane region, then

$$E = \{(x, y, z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)$$

and

$$\iiint_E f(x, y, z)\ dV = \int_z^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz\ dy\ dx.$$

Similar formula exists for a type II plane region.

A region E is of type 2 if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$$

and we have

$$\iiint_E f(x, y, z)\ dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\ dx \right] dA.$$

For a type 3 region E,

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$
$$\iiint\limits_E f(x, y, z)\ dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\ dy\right]\ dA.$$

Remark. *When we set up a triple integral, it is wise to draw two diagrams: one of the solid region E and one of its projection D.*

Remark. *The limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.*

Example 1. *Evaluate*

$$\iiint\limits_E (x - y)\ dV$$

where E is enclosed by the surfaces $z = x^2 - 1, z = 1 - x^2, y = 0,$ and $y = 2$.

Example 2. *Write five other iterated integrals that are equal to the integral*

$$\int_0^1 \int_y^1 \int_0^z f(x, y, z)\ dx\ dz\ dy.$$

It is difficult to visualize a hyper-volume under a three-variable function $f(x, y, z)$. However, if $f(x, y, z) = 1$, then

$$V(E) = \iiint\limits_E\ dV.$$

We may use triple integrals to find mass of an object with density function $\rho(x, y, z)$ that is in units of mass per unit volume:

$$m = \iiint\limits_E \rho(x, y, z)\ dV.$$

The moments of inertia about the three coordinate axes are defined as

$$I_x = \iiint\limits_E (y^2 + z^2)\rho(x, y, z)\ dV$$
$$I_y = \iiint\limits_E (x^2 + z^2)\rho(x, y, z)\ dV$$
$$I_z = \iiint\limits_E (x^2 + y^2)\rho(x, y, z)\ dV.$$

Example 3. *Find the moment of inertia for a rectangle brick with dimensions $a, b,$ and c and mass M if the center of the brick is situated at the origin and the edges are parallel to the coordinate axes. Assume that the solid has constant density k.*