15.4 Applications of Double Integrals

15.4.1 Density and Mass

Suppose the density, in units of mass per unit area, at a point (x, y) in a region D is $\rho(x, y)$, where ρ is a continuous function on D. Thus

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0. Using a Riemann sum, we obtain

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

The limit of the Riemann sum becomes an integral:

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x, y)\ dA.$$

15.4.2 Moments and Center of Mass

Recall that the mass of each small rectangle is $\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Thus the moment of this mass with respect to x axis is

$$[\rho(x_{ij}^*, y_{ij}^*)\Delta A]y_{ij}^*.$$

The moment of the entire lamina about the x-axis is

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\rho(x, y)\ dA.$$

Similarly, the moment about the y-axis is

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x, y)\ dA.$$

We define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. Physically, the lamina behaves as if its entire mass is concentrated at its center of mass. Therefore the lamina balances horizontally when supported at its center of mass.

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\ dA, \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\ dA$$

where the mass m is

$$m = \iint_D \rho(x, y)\ dA.$$

Example 1. *Find the mass and center of mass of the lamina that occupies the region D bounded by $y = x+2$ and $y = x^2$ and has density $\rho(x, y) = kx^2$.*

15.4.3 Probability

The probability density function f of a continuous random variable X has the properties that $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\ dx = 1$. The probability that X is between a and b is

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx.$$

The probability that for random variables X and Y, the point (X, Y) lies in a region D is the integral of the joint density function f:

$$P(X, Y) \in D = \iint_D f(x, y)\ dA.$$

Thus

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y)\ dy\ dx.$$

Because f is a probability density function, we have

$$f(x, y) \geq 0$$

and

$$\iint_{\mathbb{R}^2} f(x, y)\ dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1.$$

15.4.4 Expected Values

The mean of a single random variable X with density function f is

$$\mu = \int_{-\infty}^{\infty} x f(x, y)\ dx.$$

We define the X-mean and Y-mean of random variables X and Y, with joint density function f, also called the expected values of X and Y, as

$$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\ dA, \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\ dA.$$

These formulas are the same as the formulas for the center of mass, when the total mass is 1, because the total probability is 1.

Example 2. *The joint density function for a pair of random variables X and Y is*

$$f(x, y) = \begin{cases} Cx(1+y) & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.

(b) Find $P(X \leq 1, Y \leq 1)$.

(c) Find $P(X + Y \leq 1)$.