15.3 Double Integrals in Polar Coordinates

Recall the equations that relate polar and Cartesian coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

A polar rectangle is the following region:

$$R = \{(r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

We may divide a polar rectangle in a similar way that we divided a rectangle in Cartesian coordinates. The resulting small rectangles each have an area

$$\Delta A = r\ d\theta\ dr$$

If f is a continuous function on a polar rectangle R given by

$$R = \{(r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\ dr\ d\theta.$$

Example 1. *Evaluate the integral by changing to polar coordinates:*

$$\iint_D \cos\sqrt{x^2+y^2}\ dA$$

where D is the disk with center the origin and radius 2.

If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta\}$$

then

$$\iint_D f(x,y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\ dr\ d\theta.$$

Example 2. *Use a double integral to find the area of the region enclosed by both of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.*

Example 3. *Use polar coordinates to find the volume bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.*

Example 4. *Evaluate the integral*

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\ dy\ dx.$$