15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. For double integrals, we want to be able to integrate a function over any region D, not just over rectangles. We assume D is a bounded region, which means that D is enclosed in a rectangular region R. Then we define a new function F with domain R by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

We define the double integral of f over D by

$$\iint_D f(x,y)\ dA = \iint_R F(x,y)\ dA.$$

Thus we may apply all the theorems and applications that we had for integrals over a rectangle to integrals over a general region D.

We say a plane region D is of type I if D lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$. By Fubini's Theorem:

Remark. *If f is continuous on a type I region D such that*

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx.$$

Similarly, a type II region is of the form

$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Thus

$$\iint_D f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy.$$

where D is a type II region.

Example 1. *Find the volume of the solid bounded by the planes $z = x, y = x, x + y = 2$, and $z = 0$.*

Remark. *It is essential to draw a diagram that shows the region of integration. We may draw arrows that start at the lower boundary and end at the upper boundary. For a type II region we draw horizontal arrows.*

Example 2. *Evaluate the integral by reversing the order of integration.*

$$\int_0^1 \int_{x^2}^1 \sqrt{y} \sin y\ dy\ dx$$

15.2.1 Properties of Double Integrals

For the remainder of this section, we assume all the integrals exist. Double integrals have similar properties to single integrals. In particular, consider the following:

$$\iint_D [f(x,y)+g(x,y)]\ dA = \iint_D f(x,y)\ dA + \iint_D g(x,y)\ dA$$

$$\iint_D cf(x,y)\ dA = c\iint_D f(x,y)\ dA$$

$$f(x,y) \geq g(x,y) \Rightarrow \iint_D f(x,y)\ dA \geq \iint_D g(x,y)\ dA$$

$$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA$$

$$\iint_D 1\ dA = A(D)$$

$$m \leq f(x,y) \leq M \Rightarrow mA(D) \leq \iint_D f(x,y)\ dA \leq MA(D)$$

15.2.2 Average Value

Recall from the single variable functions that the average of a function $f(x)$ over an interval $[a,b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

For the two-variable functions, we have

$$f_{avg} = \frac{1}{A(R)}\iint_R f(x,y)\ dA,$$

where $A(R)$ is the area of the region R over which we integrate.