15 Multiple Integrals

15.1 Double Integrals over Rectangles

15.1.1 Review of the Definite Integral

Let's review the integral definition for a single-variable function. If $f(x)$ is defined for $a \leq x \leq b$, we divided the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$ and we chose sample point x_i^* in each i^{th} subinterval. Then the limit of the Riemann sum became the definite integral:

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

When $f(x) \geq 0$, the integral is the area under the curve f from a to b.

15.1.2 Volumes and Double Integrals

Now consider a function $f(x, y)$. First suppose $f(x, y) \geq 0$ over a closed rectangle R, with

$$R = [a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}.$$

If we let $z = f(x, y)$, then the graph of f is a surface above the rectangle R. We may estimate the volume under this surface by dividing R into sub-rectangles. We divide $[a, b]$ into m subintervals of equal width $\Delta x = \frac{b-a}{m}$ and we divide $[c, d]$ into n subintervals of equal width $\Delta y = \frac{d-c}{n}$. Thus the area of each sub-rectangle is $\Delta A = \Delta x \Delta y$. We choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij} and we approximate the part of the volume that lies above R_{ij} by the rectangular box (or column) with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. The volume of this box is base times height:

$$\Delta V_{ij} = f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Thus the total volume is

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

That is, we add all the volumes of the columns to approximate the total volume under the surface. As the numbers m and n increase, our approximation becomes more accurate. Thus

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

The above sum is called a *double Riemann sum.* For a general function $f(x, y)$ (not just nonnegative functions), we define the *double integral* of f over the rectangle R as

$$\iint\limits_R f(x, y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

and we say f is *integrable* if the limit exists. Thus if $f(x, y) \geq 0$, then the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint\limits_R f(x, y)\ dA.$$

15.1.3 Iterated Integrals

Suppose that f is an integrable function of two variables over the rectangle $R = [a, b] \times [c, d]$. The notation $\int_c^d f(x, y)dy$ means we hold x fixed and integrate $f(x, y)$ with respect to y. This is called *partial integration with respect to* y. Thus we obtain a function of x:

$$A(x) = \int_c^d f(x, y)\ dy.$$

Now we may integrate this function of x with respect to x:

$$\int_a^b A(x)\ dx = \int_a^b \left[\int_c^d f(x, y)dy\right]\ dx.$$

The above integral is called an *iterated integral*. Usually we omit the brackets and write

$$\int_a^b \int_c^d f(x, y)\ dy\ dx$$

to mean we first integrate with respect to y from c to d and then with respect to x from a to b. Similarly,

$$\int_c^d \int_a^b f(x, y)\ dx\ dy = \int_c^d \left[\int_a^b f(x, y)\ dx\right]\ dy$$

means we first integrate with respect to x, holding y fixed, from $x = a$ to $x = b$, and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$. Thus in iterated integrals, we work from the inside out.

Example 1. *Calculate the iterated integral:*

$$\int_0^1 \int_0^1 \sqrt{s + t}\ ds\ dt$$

The following theorem shows that the order of integration usually does not matter.

Theorem (Fubini's Theorem)**.** *If f is continuous on the rectangle*

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

then

$$\iint\limits_R f(x, y)\ dA = \int_c^d \int_a^b f(x, y)\ dx\ dy = \int_a^b \int_c^d f(x, y)\ dy\ dx.$$

More generally, this is true if f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Remark. *Sometimes an order of integration is easier to evaluate than the other order.*

Example 2. *Calculate the double integral:*

$$\iint\limits_R (y + xy^{-2})\ dA, \quad R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$$

Remark. *If $f(x, y) = g(x)h(y)$, then we may separate the double integral into a product of two single integrals.*

$$\iint\limits_R g(x)h(y)\ dA = \int_a^b g(x)\ dx \int_c^d h(y)\ dy, \quad R = [a, b] \times [c, d].$$

Example 3. *Calculate the double integral:*

$$\iint\limits_R \frac{\tan\theta}{\sqrt{1 - t^2}}\ dA, \quad R = \{(\theta, t) \mid 0 \leq \theta \leq \frac{\pi}{3}, 0 \leq t \leq \frac{1}{2}\}$$