14.8 Lagrange Multipliers

Lagrange devised a method to find the extreme values of a function $f(x, y, z)$, subject to constraint $g(x, y, z) = k$. Geometrically, $g(x, y) = k$ is a level curve for the function g. The optimum value of f when we select one of these level curves occurs at a point where f touches g at a point (x_0, y_0) and they have a common tangent line.

Similarly, $g(x, y, z) = k$ is a level surface for the function g. The optimum value of f when we travel on this level surface happens at a point (x_0, y_0, z_0) where f and g share the tangent plane. Thus we have:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$$

where λ is a scalar. (λ is the Greek letter lambda.)

Suppose we travel along a path C, with vector equation $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ on a level surface $g(x, y, z) = k$. Let $h(t) = f(x(t), y(t), z(t))$ be the composite function that shows the values of f along the path C. If f has an extreme value at $(x(t_0), y(t_0), z(t_0)) = (x_0, y_0, z_0)$, then h has an extreme value at $t = t_0$. Thus $h'(t_0) = 0$. By Chain Rule,

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \vec{r'}(t_0) \end{aligned}$$

Therefore $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\vec{r'}(t_0)$. Recall that $\nabla g(x_0, y_0, z_0)$ is orthogonal to $\vec{r'}(t_0)$. Therefore $\nabla f(x_0, y_0, z_0)$ is parallel to $\nabla g(x_0, y_0, z_0)$. That is, there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

This number λ is called a *Lagrange multiplier.*

Remark (The Method of Lagrange Multiplier)**.** *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:*

1. *Solve the system of equations in four unknowns x, y, z, and λ:*

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x, y, z) &= k \end{aligned}$$

2. *Evaluate f at all the points (x, y, z) that result from the previous step. The largest of these values is the maximum value of f; the smallest is the minimum value of f.*

Example 1. *Use Lagrange multipliers to find the extreme values of $f(x, y) = xe^y$ subject to constraint $x^2 + y^2 = 2$.*

Example 2. *Use Lagrange multipliers to find the extreme values of $f(x, y) = e^{xyz}$ subject to constraint $2x^2 + y^2 + z^2 = 24$.*

Example 3. *Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.*

14.8.1 Two Constraints

If there are two constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$, then $\nabla f(x_0, y_0, z_0)$ is on the plane where the vectors $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$ reside. That is, $\nabla f(x_0, y_0, z_0)$ is a linear combination of $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$:

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0),$$

where λ and μ are Lagrange multipliers. We find the extreme values by solving the system of five equations:

$$\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \\ g(x, y, z) &= k \\ h(x, y, z) &= c \end{aligned}$$