14.7 Maximum and Minimum Problems

Optimization is one of the most important problems. That is, to find the optimum value, whether it is a maximum, such as profit, or a minimum, such as cost. We may visualize maxima as peaks of mountains and minima as valleys. As in the case of single-variable functions, relative maximum values are higher than the surrounding values, and relative minimum values are lower than nearby values. The absolute maximum value is the highest of all relative maxima and the absolute minimum is the lowest point among all points.

Definition 1. *If $f(x,y) \leq f(a,b)$ for all points (x,y) near (a,b), then $f(a,b)$ is called a local maximum value. If $f(x,y) \geq f(a,b)$ for all points (x,y) near (a,b), then $f(a,b)$ is called a local minimum value. If $f(x,y) \leq f(a,b)$ for all points (x,y) in the domain of f, then $f(a,b)$ is called the absolute maximum value. If $f(x,y) \geq f(a,b)$ for all points (x,y) in the domain of f, then $f(a,b)$ is called the absolute minimum value.*

Definition 2. *A point (a,b) is called a* critical point*, or a* stationary point*, of f, if $f_x(a,b) = f_y(a,b) = 0$, or if one of these partial derivatives does not exist.*

Theorem. *If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b) = f_y(a,b) = 0$. That is, $\nabla f(a,b) = \vec{0} = \langle 0, 0 \rangle$.*

Proof. Let $g(x) = f(x,b)$. By Fermat's Theorem, $g'(a) = f_x(a,b) = 0$ at local extrema. Let $G(y) = f(a,y)$. Again, by Fermat's Theorem, $G'(b) = f_y(a,b) = 0$ at local extrema of f. □

Remark. *Just as in single-variable functions, the direction of the theorem is one-way only:*

$$(a,b) \text{ is a local extremum} \Rightarrow (a,b) \text{ is a stationary point.}$$

But the other direction is not necessarily true:

$$(a,b) \text{ is a local extremum} \not\Leftarrow (a,b) \text{ is a stationary point.}$$

Thus stationary points are candidates *for extrema points.*

The geometric interpretation is when $f_x(a,b) = f_y(a,b) = 0$, the tangent plane's equation becomes $z = z_0$, that is, the tangent plane at local extrema is horizontal.

Theorem (Second Derivative Test)**.** *Suppose the second partial derivatives of f are continuous on a disk with center (a,b). Furthermore, suppose that $f_x(a,b) = f_y(a,b) = 0$. Let*

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

$$\begin{aligned} D > 0 \text{ and } f_{xx}(a,b) > 0 &\Rightarrow f(a,b) \text{ is a local minimum.} \\ D > 0 \text{ and } f_{xx}(a,b) < 0 &\Rightarrow f(a,b) \text{ is a local maximum.} \\ D < 0 &\Rightarrow f(a,b) \text{ is a saddle point.} \end{aligned}$$

Remark. *A saddle point is neither a maximum nor a minimum. If $D = 0$, the test gives no information: anything goes! To remember the formula for D, we may write it as a determinant:*

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Example 1. *Find the local maximum and minimum values and saddle points of the function $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$.*

14.7.1 Absolute Extreme Values

We had an Extreme Value Theorem for single-variable functions, in which, continuous functions over a closed interval must have both absolute maximum and absolute minimum.

The equivalent of a closed interval in multivariable functions is a closed, bounded set.

Definition 3. *A* boundary point *of a region D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.*

A closed set *in $\mathbb{R}^2$ contains all its boundary points.*

A bounded set *in $\mathbb{R}^2$ is contained within some disk, that is, is finite.*

There is a similar theorem to the Extreme Value Theorem that we had in single-variable functions:

Theorem (Extreme Value Theorem for Functions of Two Variables)**.** *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2).*

Thus each extreme value of a function on a closed, bounded set is either a critical point of f or a boundary point of D. Similar to single-variable functions, we find absolute maximum and absolute minimum by comparing the function values at critical points and at boundary points.

Example 2. *Find the absolute maximum and minimum values of $f(x, y) = x^2 + xy + y^2 - 6y$ on the set $D = \{(x, y) \mod -3 \le x \le 3, 0 \le y \le 5\}$.*

Example 3. *Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.*