14.6 Directional Derivatives and the Gradient Vector

14.6.1 Directional Derivatives

Recall that we defined the partial derivatives of a function $z = f(x, y)$ as

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

These partial derivatives represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Now we define the *directional derivative* in *any* direction $\vec{u}$, where $\vec{u}$ is a unit vector in some direction.

Definition 1. *The* directional derivative *of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b\rangle$ is*

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Thus the previous definition of partial derivatives are special cases of this definition, with $\vec{u} = \hat{\imath} = \langle 1, 0\rangle$ or $\vec{u} = \hat{\jmath} = \langle 0, 1\rangle$.

Theorem. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b\rangle$ and*

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

14.6.2 The Gradient Vector

Notice in the above theorem that the expression for the directional derivative is a dot product.

Definition 2. *If f is a function of two variables x and y, then the* gradient *of f is the vector function ∇f defined by*

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle.$$

Thus we have

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}.$$

We may extend the definitions for functions of two variables to functions of more than two variables. Thus

$$D_u f(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

and

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle$$
$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}.$$

Example 1. *Suppose $f(x, y, z) = y^2 e^{xyz}$.*

(a) Find the gradient of f.

(b) Evaluate the gradient at the point $P(0, 1, -1)$.

(c) Find the rate of change of f at P in the direction of the vector $\vec{u} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$.

14.6.3 Maximizing the Directional Derivative

Theorem. *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.*

Proof. We have

$$D_u f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between ∇f and $\vec{u}$. The maximum value of $\cos\theta$ is 1 and occurs when $\theta = 0$ Therefore the maximum value of $D_u f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\vec{u}$ is in the same direction as ∇f. □

Example 2. *Find the maximum rate of change of $f(x, y, z) = x \ln(yz)$ at the point $(1, 2, \frac{1}{2})$ and the direction in which it occurs.*

14.6.4 Tangent Planes to Level Surfaces

Suppose we have a level surface of a function of three variables with equation

$$F(x(t), y(t), z(t)) = k.$$

If the derivatives exist, we may use the chain rule to differentiate both sides of the above equation with respect to t:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

or,

$$\nabla F \cdot \vec{r'}(t) = 0.$$

This equation says that the gradient vector at every point is orthogonal to the tangent vector at that point. We define the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. This tangent plane has the equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal to the level surface at P is the line

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

In the special case where $z = f(x, y)$ and hence $F(x, y, z) = f(x, y) - z = 0$, we would have a level surface with $k = 0$ and

$$\begin{aligned} F_x(x_0, y_0, z_0) &= f_x(x_0, y_0) \\ F_y(x_0, y_0, z_0) &= f_y(x_0, y_0) \\ F_z(x_0, y_0, z_0) &= -1 \end{aligned}$$

and the equation of the tangent plane becomes the same equation that we had before:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Example 3. *For the surface $xy + yz + zx = 5$, find equations of*

(a) the tangent plane and

(b) the normal line to the surface

at the point $(1, 2, 1)$.

14.6.5 Significance of the Gradient Vector

At a point $P(x_0, y_0, z_0)$, we have:

- By the theorem, the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f.
- ∇f is orthogonal to the level surface of f through P.
- As we move along a level surface, the function does not change. Thus it makes sense to get the fastest change in the function, we should move at a perpendicular direction.
- For two-variable functions, level surfaces become level curves. Thus gradient vector is perpendicular to a level curve.
- The rivers follow the gradient vectors as they flow from higher altitudes to lower altitudes on a contour map.

Example 4. *If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.*