14.5 The Chain Rule

Recall the chain rule for single-variable functions:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For functions of more than one variable, we have various cases of chain rule.

Theorem (Case 1 of Chain Rule)**.** *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Remark. *Notice the similarity of the chain rule to the definition of the differential:*

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Example 1. *Use chain rule to find dw/dt when $w = \ln\sqrt{x^2 + y^2 + z^2}$ and $x = \sin t, y = \cos t, z = \tan t$.*

Theorem (Case 2 of Chain Rule)**.** *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of s and t and*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

In case 2, s and t are the independent variables, x and y are the intermediate variables, and z is the dependent variable.

Remark. *To remember the chain rule, we may draw a tree diagram. We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t. On each branch, we write the corresponding partial derivative. For example, to find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

Example 2. *Use the chain rule to find $\partial z/\partial s$ and $\partial z/\partial t$ when $z = \tan^{-1}(x^2 + y^2), x = s \ln t, y = te^s$.*

Theorem (General Version of Chain Rule)**.** *Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Example 3. *Use a tree diagram to write out the Chain Rule for $R = F(t, u)$, where $t = t(w, x, y, z), u = u(w, x, y, z)$. Assume all functions are differentiable.*

Example 4. *The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?*

Example 5. *If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

Example 6. *If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find*

(a) $\partial z/\partial r$

(b) $\partial z/\partial \theta$

(c) $\partial^2 z/\partial r \partial \theta$.

14.5.1 Implicit Differentiation

If $F(x, y) = 0$ defines y implicitly as a function of x, we may differentiate implicitly with respect to x:

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Since $dx/dx = 1$, we may solve for dy/dx to obtain

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}.$$

Similarly, if $F(x, y, z) = 0$ defines z implicitly as a function of (x, y), we may differentiate both sides of the equation with respect to x:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

Since $\partial x/\partial x = 1$ and $\partial y/\partial x = 0$, we may solve for $\partial z/\partial x$ to obtain

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z}.$$