14.4 Tangent Planes and Linear Approximations

Recall that we may approximate a function near a point with its tangent line. In this section we extend this idea to functions of more than one variable. For a function of two variables, we may approximate the function with the linear equation of a tangent plane near the tangent point.

14.4.1 Tangent Planes

Recall for a function of one variable, we had the equation of the tangent line at the point (x_0, y_0) as

$$y - y_0 = f'(x_0)(x - x_0)$$

Similarly, the equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

A tangent plane contains all possible tangent lines at the tangent point to curves that lie on the surface and pass through the tangent point. In particular, the tangent plane is made from the tangent lines to the intersection curves between a surface and planes $x = x_0$ and $y = y_0$.

Example 1. *Find the equation of the tangent plane to the surface* $z = \ln(x - 2y)$ *at the point* $(3, 1, 0)$.

14.4.2 Linear Approximations

A tangent plane is a good approximation to the surface $z = f(x, y)$ near the tangent point. We say the function

$$L(x, y) = ax + by + c$$

is the *linearization* of the function $f(x, y)$ at point (x, y) when L is the equation of the tangent plane to f at (x, y). Therefore, near the tangent point, we have

$$f(x, y) \approx L(x, y)$$

This approximation is called the linear (tangent plane) approximation of f at a point. To derive the equation of the tangent plane, we would need to partially differentiate the function f.

Theorem. *If the partial derivatives* f_x *and* f_y *exist near* (a, b) *and are continuous at* (a, b)*, then* f *is differentiable at* (a, b).

Example 2. *Explain why the function*

$$f(x, y) = 1 - xy\cos\pi y$$

is differentiable at $(1, 1)$*. Then find the linear approximation of the function at the point* $(1, 1)$ *and use it to approximate* $f(1.02, 0.97)$.

14.4.3 Differentials

Let's start with a review of the notion of a differential in functions of one variable. We defined the differential dx as an independent variable that can be any real number. Then the differential dy was defined as

$$dy = f'(x)dx.$$

If Δy is the actual change in y along the curve $y = f(x)$, then dy is the change along the tangent line to the curve, going from x to $x + dx$.

Extending these ideas to a function of two variables, we define the differential

$$df = dz = f_x(x, y)dx + f_y(x, y)dy,$$

where dx and dy are independent variables and dz is the total differential. Here, Δz represents the actual change in $z = f(x, y)$, whereas dz is the change on the tangent plane, going from (a, b) to $(a + dx, b + dy)$. If $\Delta x = dx = x - a$ and $\Delta y = dy = y - b$, then

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

and hence we may write the linear approximation as

$$f(x, y) \approx f(a, b) + dz.$$

Example 3. *The wind-chill index is modeled by the function*

$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the temperature ($^\circ C$) and v is the wind speed (km/h). When $T = -15^\circ C$ and $v = 30$ km/h, by how much would you expect the apparent temperature W to drop

a) if the actual temperature decreases by $1^\circ C$?

b) if the wind speed increases by 1 km/h?

14.4.4 Functions of Three or More Variables

For functions of more than two variables, we define the linearization, linear approximation, and differential in a similar manner to functions of two variables.

Example 4. *Find the differential of the function $L = xze^{-y^2-z^2}$.*